Exploratory Studies of Joined Wing Aeroelasticity

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Interest in Joined Wing configurations has been growing recently especially with the appearance of new possible applications such as the sensorcraft and advanced aerial tanker. The aeroelastic behavior of Joined Wings is the focus of this work. Particular attention is given to the effect of structural nonlinearity on the divergence and linearized flutter predictions for such configurations. In a typical flight condition, significant compressive loads are present in the rear wing (which supports the main wing) and its effective stiffness (linear + geometric) varies. As a consequence, the aeroelastic behavior of the wing system changes significantly and a nonlinear structural analysis is required. The paper reports the results of a study based on linear aerodynamic theory and a nonlinear Updated Lagrangian Formulation for the structural part. The goal is to demonstrate an approach to the evaluation of aeroelastic characteristics in the early design stages.

Nomenclature

$1^S, 2^S, 3^S, 4^S$ Points that identify the wing segment $S$

$\alpha$ Angle of attack

$a_{km}$ Influence coefficient

$P_c^k$ Control point

$P_L^k$ Load point

$V_\infty$ Freestream velocity

$\rho_\infty$ Air density

$\Gamma$ Circulation

$x, y, z$ Coordinate system

$x_1^S, y_1^S, z_1^S$ Local coordinate system on surface $S$

$L_{ref}$ Reference aerodynamic load

$P_{ref}$ Reference non-aerodynamic load

$N_{step}$ Number of load steps

$\lambda$ Generic load step

$n$ Generic iteration

$N$ Number of panels

$N^S$ Number of panels on surface $S$

$x_2^S, y_2^S, z_2^S$ Global coordinates of the origin of the local system on surface $S$

$Z^S_{loc}$ Local $z$ coordinate of the $i^{th}$ structural node on surface $S$

$x_1^S_{loc}, y_1^S_{loc}$ Local coordinates (on the reference plane) of the $i^{th}$ structural node on surface $S$

$N^S_{n}$ Number of structural nodes on surface $S$

$a_0^S, a_1^S, a_2^S, F^S_j$ Coefficients of the spline used in surface $S$

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Local $z$ coordinate of the $i^{th}$ control point on surface $S$

Local coordinates (on the reference plane) of the $i^{th}$ control point on surface $S$

Laplace variable

Reference half chord

$j^{th}$ eigenvalue

Real part of the $j^{th}$ eigenvalue

Complex part of the $j^{th}$ eigenvalue

Circular frequency

Frequency

Reduced frequency

Maximum reduced frequency

Maximum circular frequency

Geometrical dimension (different for each analyzed case)

Elastic modulus

Poisson’s ratio

Spring stiffness (linear case)

Spring stiffness (nonlinear case)

Elastic deformation

Contribution of the spring to the tangent stiffness matrix

Reference aerodynamic speed

Reference aerodynamic speed (divergence speed of a swept-forward wing)

Sweep angle

Chord

Applied uniform pressure

Non-dimensional pressure

Thickness of the plate

Displacement in the $z$ direction

Dynamic pressure

Angular coefficient of lift

Geometric stiffness matrix

Induced velocity on the panel $k$ (at point $P_k^S$) induced by the vortices of the panel $m$

Small rigid rotation vector

Normal to the panel $k$

Unit vector ($x$ direction)

Unit vector ($y$ direction)

Unit vector ($z$ direction)

Vector which contain the vorticity of all panels

Matrix of the influence coefficients

Right hand side vector used in the imposition of the Wall Tangency Condition

Vector containing all lifting forces

Transformation matrix

Translation displacements (no rotations) of the structural nodes on surface $S$

Displacements of all structural nodes

Transformation matrices

Vector containing the global coordinates of all structural nodes

Vector containing the coordinates of the structural nodes on surface $S$

Vector containing the updated coordinates of the structural nodes on surface $S$

Vector containing the coordinates of the origin of the local coordinate system
\(e^S\) Rotation matrix
\(E^S\) Transformation matrix obtained using \(e^S\)
\(Z^S_{\text{loc}}\) Vector containing the coordinates \(Z^S_{\text{loc}}\)
\(K^S, K^S\) Matrices used in the spline derivations
\(R^S, G^S\) Matrices used in the spline derivations
\(D^S, S^S\) Matrices used in the spline derivations
\(F^S\) Vector containing the coefficients of the spline used in the wing surface \(S\)
\(Z^S_{\text{loc}}\) Vector obtained from \(Z^S_{\text{loc}}\) by adding three rows of zeros
\(I^S\) Transformation matrix
\(Z^S\) Vector containing the local \(z\) coordinates of the control points on surface \(S\)
\(A\) Constant vector used in the definition of the aerodynamic loads
\(B\) Constant matrix used in the definition of the aerodynamic loads
\(C\) Aerodynamic tangent matrix
\(L_{\text{str}}\) Aerodynamic loads calculated not considering the aerodynamic tangent matrix
\(P_{\text{str}}\) Current non-aerodynamic loads
\(P_{\text{ext}}\) Applied non-aerodynamic loads
\(F_{\text{int}}\) Internal forces
\(P_{\text{unb}}\) Unbalanced loads
\(K_T\) Structural tangent matrix
\(M\) Mass matrix
\(A\) Generalized aerodynamic matrix
\(M\) Generalized mass matrix
\(K_T\) Generalized stiffness matrix
\(A_0, A_i, A_2\) Roger matrices
\(A_3, A_4\) Roger matrices
\(M^*\) Generalized mass matrix of the coupled system
\(K^*\) Generalized stiffness matrix of the coupled system
\(C^*\) Aerodynamic damping matrix
\(q\) Generalized coordinates
\(X_1, X_2\) State vector
\(X_3, X_4\) State vector
\(X\) State vector with displacements, speeds and aerodynamic states
\(I\) Identity matrix
\(U, V\) System matrices

**Subscript**
- IP In-plane
- OUT Out-of-plane

**Superscript**
- \(S\) Referred to the surface \(S\)
- \(T\) The transpose is calculated
- \(-1\) The inverse of the matrix is calculated
- \(\text{step } \lambda \text{ iter } n\) Referred to step \(\lambda\) and iteration \(n\)

**Acronym**
- JW Joined Wing
- WTC Wall Tangency Condition
- IPS Infinite Plate Spline
I. Introduction

The Joined Wing (JW) configuration has been the subject of aerodynamic, structural, and design optimization studies for almost 30 years. Its aeroelastic behavior, however, and its effect on design are still not completely understood, and only a very small number of exploratory studies addressed JW aeroelasticity in a satisfactory way.

With Joined-Wings, the substantial tail - rear wing surface can be under significant in-plane compression and the inboard part of the wing is under in-plane tension. Structural geometric nonlinearity becomes important and can affect aeroelastic behavior by leading to divergence of the rear wing, changing natural frequencies under maneuver conditions with the resulting impact on flutter, or leading to interactions between static and dynamic instabilities similar to those found in panel flutter. Aeroelastic investigation of geometrically nonlinear lifting surfaces in the past few years covers high-aspect ratio wings of high-altitude long-endurance aircraft (HALE), strut-braced wings, wind tunnel models of delta and beam-like wings, and joined-wing configurations. The proposed paper will present a modeling capability for investigating the fundamental problems of JW aeroelastic behavior. The capability consists of a dedicated nonlinear finite element code for 3D plate assembly configurations, combined with a Vortex Lattice aerodynamic simulation allowing for aerodynamic mesh deformation, and a linear unsteady aerodynamic code for small-perturbation stability and response analyses.

Results of steady and unsteady linear and nonlinear aeroelastic simulations for a JW configuration will then presented and discussed.

II. Nonlinear Structural Model

The geometrically nonlinear structural model is built using flat triangular elements. The tangent stiffness matrix is built adding the linear elastic stiffness matrix and the geometric stiffness matrix. The geometric stiffness matrix is built applying the load perturbation method: the gradient (with respect to the coordinates) of the nodal force vector (when the stresses are considered fixed) is calculated. The geometric stiffness matrix is calculated adding 4 matrices:

\[
[K^e_{\text{mem}}]_{\text{TOTAL}} = [K^e_{\text{mem}}]_{\text{IP}} + [K^e_{\text{plate}}]_{\text{IP}} + [K^e_{\text{mem}}]_{\text{OUT}} + [K^e_{\text{plate}}]_{\text{OUT}}
\]  

The matrix \([K^e_{\text{mem}}]_{\text{IP}}\), representing the in-plane contribution of the plane stress triangular element (CST), is obtained taking the gradient of the nodal forces. The matrix \([K^e_{\text{plate}}]_{\text{IP}}\), representing the in-plane contribution of the triangular plate bending element, is calculated using a similar approach applied to the triangular element based on the Discrete Kirchoff Theory (DKT). The matrix \([K^e_{\text{mem}}]_{\text{OUT}}\) representing the out-of-plane contribution of the membrane, is calculated considering the change of a vector force which is subjected to a small rigid rotation vector \(\omega\). Similar approach is conducted in order to calculate the matrix \([K^e_{\text{plate}}]_{\text{OUT}}\) which represents the out-of-plane contribution of the plate.

A particular procedure is then used in order to remove the rigid body motion and calculate the unbalanced load as the analysis (Newton Raphson) progresses.

The original work of Refs. 17 and 18 has been improved adding new features. In particular, a lumped mass matrix calculation was added. The capability to perform free vibration analysis under load was added as well. Moreover, the Newmark time integration method was added to carry out nonlinear time-dependent simulations (transient analyses). A pre-processor has been created, allowing the reading of mathematical models generated for MSC-NASTRAN. Finally, the code has been integrated with a linear vortex lattice aerodynamic code written for the purpose to perform static aeroelastic analyses (with geometric nonlinearities)
ties). The calculation of the aerodynamic loads, the transferring of the loads from the aerodynamic mesh to the structural mesh have been added (for more details see the next sections). The computation of inertial loads was also added.

III. Vortex Lattice Formulation

The geometry of a generic non-planar wing system is reported in figure 1. The velocity $V_\infty$ is assumed directed along $+x$. The wing is discretized using wing segments (called also wing surfaces in the paper). Consider the wing segment $S$ (see figure 2). Suppose that each surface has zero twist. Let the 4 nodes which characterize the surface be called with $1^S$, $2^S$, $3^S$ and $4^S$. These points are chosen using the following logic:

$$x_1^S > x_2^S; \quad x_2^S > x_3^S$$

Consider now an element (for example the element $k$, see also figure 3). Called with $V_k^m$ the induced velocity on the panel $k$ (at point $P_k$) induced by the vortices of the panel $m$, the influence coefficients are defined as $a_{km}^\Gamma = [V_k^m]^T \cdot n_k$, where $n_k$ is the normal to the panel $k$ at the control point at point $P_k$. In the case of symmetry conditions (for example the plane $x-z$) the induced velocity is differently calculated.

The Wall Tangency Condition has to be imposed for all panels of all surfaces. Considering the assumption that the freestream velocity is directed along $+x$, for the panel $k$ the Wall Tangency Condition is:

$$a_1^\Gamma \Gamma_1 + a_2^\Gamma \Gamma_2 + a_3^\Gamma \Gamma_3 + ... + a_N^\Gamma \Gamma_N + V_\infty i^T \cdot n_k = 0$$

where $\Gamma_1, \Gamma_2...\Gamma_N$ are the circulations. Considering all $N$ panels, the WTC can be written in a matrix form as

$$A^\Gamma \cdot \Gamma = -V_\infty H_{\text{RHS}}$$

where

$$H_{\text{RHS}} = \begin{bmatrix} i^T \cdot n_1 & i^T \cdot n_2 & i^T \cdot n_3 & ... & i^T \cdot n_N \end{bmatrix}^T$$

The modulus of the aerodynamic lifting force acting on panel $k$ is

$$L_k = \rho_\infty V_\infty \Gamma_k \Delta b_k = \rho_\infty V_\infty \Gamma_k \left| \mathbf{i} \times \mathbf{r}_{\Gamma 4^k}^{-1} \mathbf{i} \right|$$

Notice that the direction of the lift is parallel to the vector $\mathbf{i}_L = \mathbf{i} \times \mathbf{r}_{\Gamma 4^k}^{-1}$ (the lifting force has no component in the $x$ direction). Operating similarly for all aerodynamic panels, it is possible to organize the aerodynamic lifting forces in a vector as follows:

$$L = \rho_\infty V_\infty I_{\Gamma} \cdot \Gamma = -\rho_\infty V_\infty^2 I_{\Gamma} \cdot \left[ A^\Gamma \right]^{-1} H_{\text{RHS}}$$

where the transformation matrix $I_{\Gamma}$ depends on the orientation of the panels in space.
IV. Calculation of the Aerodynamic Forces

The vortices used in the vortex lattice formulation are not moved even if the structure deforms. This assumption is valid if the displacements are not too large. However, even if the vortices are kept fixed, their intensity changes because the Wall Tangency Condition has to be imposed using the new orientation of the elements in the space. Relations 4 and 7 are still valid, but it has to be clear that the matrix $A_{\Gamma}$ does not change and the matrix $I_{\Gamma}$ does not change as well (this because the aerodynamic force direction depends on the external product between the vectors that identify the vortices and the vortices do not move under the hypothesis of linear aerodynamic theory). The vorticity vector $\Gamma$ does change during the deformation process and reason is that the Wall Tangency Condition has to imposed considering the new directions of the normals to the aerodynamic panels at the control points. Thus, the vectors $H_{RHS}$ and $\Gamma$ do change even if the aerodynamic formulation is linear. The structure includes geometric nonlinearities. Therefore, the aerodynamic pressure cannot be applied at once: load steps have to be defined. For example, if $N_{\text{step}}$ is the number of steps that are considered (it has to be relatively large in order to achieve the convergence), the reference aerodynamic load is:

$$L_{\text{ref}} = -\frac{\rho \infty V_{\infty}^2}{N_{\text{step}}}$$

At the beginning of the generic iteration $n$ corresponding to the load step $\lambda$, the aerodynamic loads will be

$$L^{\text{step } \lambda \text{ iter } n} = \lambda L_{\text{ref}} I_{\Gamma} \left[A_{\Gamma}\right]^{-1} H^{\text{step } \lambda \text{ iter } n}_{\text{RHS}}$$

(9)

The vector $H^{\text{step } \lambda \text{ iter } n}_{\text{RHS}}$ depends on the initial coordinates of the control points at the iteration $n$ and, also, it has a contribution from the displacements referred to the coordinates at the beginning of the iteration $n$. The matrix that multiplies the displacements has the meaning of aerodynamic tangent matrix.
In order to impose the Wall Tangency Condition, the derivatives of the structural shape have to be calculated. However, the shape is known only at the structural points and the derivatives calculated at the control points are not known. Thus, an interpolation is necessary and the Infinite Plate Spline method is applied. The method is applied for each wing segment into which the wing is divided (notice that the wing segments are not the aerodynamic panels, but only macro elements in which the wing is divided). The undeformed initial configuration will represent the reference plane used in the IPS approximation. In general the wing system can be non-planar and this approach is general and can be used to analyze wings generally positioned in the space (such as Joined Wings). Notice that different wing segments have different reference planes.

**A. Displacements and Coordinate Transformation in a Generic Wing Segment**

Calling $I^S$ the matrix that allows to transform the global displacements of the structural nodes to the displacements of the structural nodes included in the surface $S$ and calling $I^S_d$ the matrix which allows to calculate the displacements (which do not include the rotations) of the wing segment $S$, it is possible to express the translational displacement vector $u^{S\text{step }\lambda\text{iter }n}$ (expressed in global coordinates) using the relation:

$$u^{S\text{step }\lambda\text{iter }n} = I^S_d \cdot I^S \cdot u^{\text{step }\lambda\text{iter }n}$$

The vector $u^{\text{step }\lambda\text{iter }n}$ contains all displacements and rotations of all structural nodes of the structure (i.e., all wing segments are included). Moreover, $u^{\text{step }\lambda\text{iter }n}$ is referred to the previous coordinates (Updated Lagrangian Formulation). That implies that the vector $u^{\text{step }\lambda\text{iter }n}$ is not referred to the initial undeformed configuration. Calling $x^{\text{step }\lambda\text{iter }n}$ the vector of the global coordinates of all structural nodes at the beginning of the $n^{th}$ iteration and calling $I^S_x$ an opportune transformation matrix, the vector $x^{S\text{step }\lambda\text{iter }n}$ containing the global coordinates of the structural nodes on wing segment $S$ is:

$$x^{S\text{step }\lambda\text{iter }n} = I^S_x \cdot x^{\text{step }\lambda\text{iter }n}$$

The matrices $I^S_x$, $I^S_d$ and $I^S$ do not depend either on the displacements nor on the iteration. They depend only on the wing segment that is considered. Adding the translational displacements of the nodes in the wing segments, the vector $x^{\text{step }\lambda\text{iter }n}$ becomes:

$$x^{\text{step }\lambda\text{iter }n} = I^S_x \cdot x^{\text{step }\lambda\text{iter }n}$$

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**Figure 3. Element $k$. Geometry and notation.**

- $P_C^k$ = Control point (where the WTC is imposed)
- $P_L^k$ = Lift point (where the lift is applied)
- $r_{4^k-1^k}$ = Vector which connects the points $4^k$ and $1^k$ (origin in $1^k$)
segment $S$ to the global coordinates $x^{S\text{step}\lambda\text{iter}\,n}$ of the same nodes at the beginning of the iteration $n$, it is possible to express the coordinates $X^{S\text{step}\lambda\text{iter}\,n}$ at the end of the iteration $n$ as

$$X^{S\text{step}\lambda\text{iter}\,n} = x^{S\text{step}\lambda\text{iter}\,n} + u_d^{S\text{step}\lambda\text{iter}\,n} = I_x^S \cdot x^{S\text{step}\lambda\text{iter}\,n} + I_d^S \cdot u^{S\text{step}\lambda\text{iter}\,n}$$  \hspace{1cm} (12)

In order to approximate the actual shape with a set of known functions, it is necessary to define a coordinate system on the wing surface $S$ (see figure 4). The matrix which allows to change the coordinates system (from global to local) is indicated with $e^S$. The coordinates (in the original undeformed configuration) of the local coordinate system origin (point 2) is considered. Using that assumption, it can be inferred that the original local coordinates of the points on the wing segment $S$ expressed in the local coordinate system are determined subtracting the global coordinates of the point $2^S$ (considered in the original undeformed condition) and multiplying the result by the matrix $e^S$. Introducing the vector $x_{2s}$ (which has dimension $3N_n^S \times 1$), $N_n^S$ is the number of structural nodes on wing segment $S$)

$$x_{2s} = \begin{bmatrix} x_{2s} & y_{2s} & z_{2s} & \ldots & x_{2s} & y_{2s} & z_{2s} \end{bmatrix}^T$$  \hspace{1cm} (13)

and the matrix $E^S$ (which has in the diagonal the matrix $e^S$ repeated $N_n^S$ times), it is possible to write:

$$X^{S\text{step}\lambda\text{iter}\,n}_{\text{loc}} = E^S \cdot \left[ X^{S\text{step}\lambda\text{iter}\,n} - x_{2s} \right] = E^S \cdot \left[ I_x^S \cdot x^{S\text{step}\lambda\text{iter}\,n} + I_d^S \cdot u^{S\text{step}\lambda\text{iter}\,n} - x_{2s} \right]$$  \hspace{1cm} (14)

**B. WTC Imposition Using Infinite Plate Splines**

In order to calculate the aerodynamic incidence in each panel of the wing surface $S$, it is necessary to calculate the derivative of the shape with respect to the local $x$ axis $x^S$ (see figure 4). This operation is performed using the Infinite Plate Splines method (IPS).

In order to do that, the local coordinates $Z^{S\text{step}\lambda\text{iter}\,n}_{\text{loc}}$ in the $z_2$ direction have to be isolated. Calling $I_z^S$ the matrix which allows to isolate the local $z^S$ coordinates, it is possible to write:

$$Z^{S\text{step}\lambda\text{iter}\,n}_{\text{loc}} = I_z^S \cdot X^{S\text{step}\lambda\text{iter}\,n}_{\text{loc}} = I_z^S \cdot E^S \cdot \left[ I_x^S \cdot x^{S\text{step}\lambda\text{iter}\,n} + I_d^S \cdot I^S \cdot u^{S\text{step}\lambda\text{iter}\,n} - x_{2s} \right]$$  \hspace{1cm} (15)

Suppose that the $i^{\text{th}}$ structural point on wing segment $S$ is considered. The local $z$ coordinate of the point $i$ will be $Z^{S\text{step}\lambda\text{iter}\,n}_{i\text{loc}}$. If the deformation is not too large and the linear aerodynamic theory is applied, it is possible to assume that the original local coordinates $x^{S\text{iter}\,i}_{\text{loc}}, y^{S\text{iter}\,i}_{\text{loc}}$ of the point $i$ do not change during the deformation process. With this assumption it is not necessary any projection of the structural nodes in the reference plane of wing surface $S$. Practically it is assumed that the projection is always corresponding to the initial position of the structural point that is considered. Using that assumption, it can be inferred that

$$Z^{S\text{step}\lambda\text{iter}\,n}_{i\text{loc}} = Z^{S\text{step}\lambda\text{iter}\,n}_{i\text{loc}} \left( x^{S\text{iter}\,i}_{\text{loc}}, y^{S\text{iter}\,i}_{\text{loc}} \right)$$  \hspace{1cm} (16)
where

\[
(r_{ij}^S)^2 = (x_{i\text{ loc}}^S - x_{j\text{ loc}}^S)^2 + (y_{i\text{ loc}}^S - y_{j\text{ loc}}^S)^2
\]

Equation (17) can be rewritten introducing the matrix \(K^S\) defined as

\[
K_{ij}^S = (r_{ij}^S)^2 \ln(r_{ij}^S)^2
\]

Thus,

\[
Z_{i\text{ loc}}^S \lambda_{\text{ iter}} n = a_0 S \lambda_{\text{ iter}} n + a_1 S \lambda_{\text{ iter}} n x_{i\text{ loc}}^S + a_2 S \lambda_{\text{ iter}} n y_{i\text{ loc}}^S + \sum_{j=1}^{N_n^S} F_j S \lambda_{\text{ iter}} n (r_{ij}^S)^2 \ln(r_{ij}^S)^2
\]

Also the following conditions have to be satisfied:

\[
\sum_{j=1}^{N_n^S} F_j S \lambda_{\text{ iter}} n = 0
\]

\[
\sum_{j=1}^{N_n^S} F_j S \lambda_{\text{ iter}} n x_{j\text{ loc}}^S = 0
\]

\[
\sum_{j=1}^{N_n^S} F_j S \lambda_{\text{ iter}} n y_{j\text{ loc}}^S = 0
\]

Equations (21) can be combined in a compact form. Setting (notice that \(Z_{i\text{ loc}}^S \lambda_{\text{ iter}} n\) is coincident with \(Z_{i\text{ loc}}^S \lambda_{\text{ iter}} n\) except for the fact that three rows of zeros have been added)

\[
Z_{\text{ loc}}^S \lambda_{\text{ iter}} n = \begin{bmatrix}
0 & 0 & 0 & Z_{1\text{ loc}}^S \lambda_{\text{ iter}} n & Z_{2\text{ loc}}^S \lambda_{\text{ iter}} n & Z_{3\text{ loc}}^S \lambda_{\text{ iter}} n & ... & Z_{N_n^S} S \lambda_{\text{ iter}} n
\end{bmatrix}^T
\]

\[
F_{\text{ loc}}^S \lambda_{\text{ iter}} n = \begin{bmatrix}
a_0 S \lambda_{\text{ iter}} n & a_1 S \lambda_{\text{ iter}} n & a_2 S \lambda_{\text{ iter}} n & F_1 S \lambda_{\text{ iter}} n & F_2 S \lambda_{\text{ iter}} n & F_3 S \lambda_{\text{ iter}} n & ... & F_{N_n^S} S \lambda_{\text{ iter}} n
\end{bmatrix}^T
\]

It is possible to obtain the relation

\[
Z_{\text{ loc}}^S \lambda_{\text{ iter}} n = \begin{bmatrix}
0 & R^S
\end{bmatrix}^T F_{\text{ loc}}^S \lambda_{\text{ iter}} n = G^S F_{\text{ loc}}^S \lambda_{\text{ iter }} n
\]

Inverting equation (23) it is possible to find the \(N_n^S + 3\) unknowns represented by the components of the vector \(F_{\text{ loc}}^S \lambda_{\text{ iter }} n\):

\[
F_{\text{ loc}}^S \lambda_{\text{ iter }} n = \begin{bmatrix}G^S\end{bmatrix}^{-1} Z_{\text{ loc}}^S \lambda_{\text{ iter}} n
\]

Now the coefficients that have to be used for the spline are known.

The Wall Tangency Condition is imposed at the control points. Let the local coordinates (in the reference

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plane) of the \(i^{th}\) control point be indicated with \(\mathcal{X}_{i}^{S}_{\text{loc}}\) and \(\mathcal{Y}_{i}^{S}_{\text{loc}}\). As for the structural points, it is assumed that their projection on the reference plane of wing segment \(S\) does not change. The coordinate \(Z_{i}^{S \text{ step } \lambda \text{ iter } n}_{\text{loc}}\) in the direction of \(z^{S}\) of the \(i^{th}\) control point will be calculated using the equation of the spline:

\[
Z_{i}^{S \text{ step } \lambda \text{ iter } n}_{\text{loc}} (\mathcal{X}_{i}^{S}_{\text{loc}}, \mathcal{Y}_{i}^{S}_{\text{loc}}) = a_{0}^{S \text{ step } \lambda \text{ iter } n} + a_{1}^{S \text{ step } \lambda \text{ iter } n} \mathcal{X}_{i}^{S}_{\text{loc}} + a_{2}^{S \text{ step } \lambda \text{ iter } n} \mathcal{Y}_{i}^{S}_{\text{loc}} + \sum_{j=1}^{N_{S}} F_{j}^{S \text{ step } \lambda \text{ iter } n} K_{ij}^{S} \tag{25}
\]

where

\[
K_{ij}^{S} = (R_{ij}^{S})^{2} \ln(R_{ij}^{S})^{2} \tag{26}
\]

and

\[
(R_{ij}^{S})^{2} = (\mathcal{X}_{i}^{S}_{\text{loc}} - x_{j}^{S}_{\text{loc}})^{2} + (\mathcal{Y}_{i}^{S}_{\text{loc}} - y_{j}^{S}_{\text{loc}})^{2} \tag{27}
\]

The number of control points is, of course, the same as the number of aerodynamic panels \(N^{S}\) in which the wing segment \(S\) is divided. In order to calculate the vector \(H_{\text{RHS}}^{S \text{ step } \lambda \text{ iter } n}\) the derivatives with respect to \(x^{S}\) are required (see Figure 1). Therefore, it is necessary to differentiate the spline equation with respect to \(x^{S}\) and calculate the result in the local coordinates of the control points. Observing that

\[
\frac{dK_{ij}^{S}}{dx^{S}} = 2 \left( \mathcal{X}_{i}^{S}_{\text{loc}} - x_{j}^{S}_{\text{loc}} \right) \left[ \ln(R_{ij}^{S})^{2} + 1 \right] \tag{28}
\]

the result of the derivation is:

\[
\frac{dZ_{i}^{S \text{ step } \lambda \text{ iter } n}_{\text{loc}}}{dx^{S}} = D^{S} F^{S \text{ step } \lambda \text{ iter } n} \tag{29}
\]

The matrix \(D^{S}\) contains the terms \(\frac{dK_{ij}^{S}}{dx^{S}}\) and its explicit form is omitted. Using equation 24, equation 29 can be written as

\[
\frac{dZ_{i}^{S \text{ step } \lambda \text{ iter } n}_{\text{loc}}}{dx^{S}} = D^{S} F^{S \text{ step } \lambda \text{ iter } n} = D^{S} \left[ G^{S} \right]^{-1} Z_{i}^{S \text{ step } \lambda \text{ iter } n}_{\text{loc}} \tag{30}
\]

Observing that the first three rows of \(Z_{i}^{S \text{ step } \lambda \text{ iter } n}_{\text{loc}}\) are zeros, it is possible to eliminate the first three columns of the matrix \(G^{S}\) without changing the result. Defining \(S^{S}\), the matrix \(G^{S}\) with the first three columns eliminated, and defining \(Z_{i}^{S \text{ step } \lambda \text{ iter } n}_{\text{loc}}\), the vector \(Z_{i}^{S \text{ step } \lambda \text{ iter } n}_{\text{loc}}\) with the first three columns, equation 30 can be rewritten as

\[
\frac{dZ_{i}^{S \text{ step } \lambda \text{ iter } n}_{\text{loc}}}{dx^{S}} = D^{S} S^{S} Z_{i}^{S \text{ step } \lambda \text{ iter } n}_{\text{loc}} \tag{31}
\]

Using equation 15, relation 14 can be written as

\[
\frac{dZ_{i}^{S \text{ step } \lambda \text{ iter } n}_{\text{loc}}}{dx^{S}} = D^{S} S^{S} I^{S}_{z} \cdot E^{S} \cdot \left[ I^{S}_{z} \cdot x^{\text{ step } \lambda \text{ iter } n}_{\text{loc}} + I^{S}_{d} \cdot I^{S} \cdot u^{\text{ step } \lambda \text{ iter } n}_{\text{loc}} - x^{S}_{z} \right] \tag{32}
\]

Consider now the \(k^{th}\) panel on the surface \(S\). The corresponding derivative calculated at the control point of the \(k^{th}\) panel is \(\frac{dZ_{k}^{S \text{ step } \lambda \text{ iter } n}_{\text{loc}}}{dx^{S}}\). Using figure 5:

\[
i^{T} \cdot n_{k} = \cos \left( \frac{\pi}{2} - \alpha^{k} \right) = \sin \alpha^{k} = \sin \left( \pi^{k} + \beta^{k} \right) = \sin \pi^{k} \cos \beta^{k} + \cos \pi^{k} \sin \beta^{k} \tag{33}\]

where \(\pi^{k}\) is the angle of attack of the panel \(k\) in the initial undeformed condition. Under the assumption of small angle of attack (it is consistent with the other assumptions), it is easy to see that

\[
\cos \pi^{k} \approx \cos \beta^{k} \approx 1
\]

\[
\sin \beta^{k} \approx \tan \beta^{k} = - \tan \left( \pi - \beta^{k} \right) = - \frac{dZ_{k}^{S \text{ step } \lambda \text{ iter } n}_{\text{loc}}}{dx^{S}} \tag{34}\]

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Substituting these quantities into equation (33) and considering that \( \sin \alpha^k = \vec{t}^T \cdot \vec{m}_k \) is the initial RHS of the vortex lattice problem (so it is known and calculated only once using the geometry of the wings):

\[
\vec{t}^T \cdot \vec{m}_k = \vec{t}^T \cdot \vec{n}_k = \frac{dZ_{loc}^{\text{step } \lambda \text{ iter } n}}{dx^S} \quad (35)
\]

Now all quantities are known, and it is possible to find \( H_{RHS}^{S \text{ step } \lambda \text{ iter } n} \) putting all values, relative to the control points of all panels, in a vector (equation (5) is applied to the panels on surface \( S \)). Thus,

\[
H_{RHS}^{S \text{ step } \lambda \text{ iter } n} = H_{RHS}^S - \frac{dZ_{loc}^{S \text{ step } \lambda \text{ iter } n}}{dx^S} \quad (36)
\]

Using equation (32), \( H_{RHS}^{S \text{ step } \lambda \text{ iter } n} \) can be written as

\[
H_{RHS}^{S \text{ step } \lambda \text{ iter } n} = H_{RHS}^S - D^S S^S \left[ I^S_x \cdot \vec{x}^{\text{step } \lambda \text{ iter } n} + I^S_d \cdot \vec{u}^{\text{step } \lambda \text{ iter } n} - x^{2S}_x \right] \quad (37)
\]

Figure 5. Wing surface \( S \). Meaning of \( dZ_{loc}^{\text{step } \lambda \text{ iter } n} \).
The previous equation can be written in a compact form:

\[ H_{\text{RHS}}^S \text{step} \lambda \text{iter} n = a^S + b^S x^\text{step} \lambda \text{iter} n + c^S u^\text{step} \lambda \text{iter} n \]  

(38)

where

\[ a^S = \Pi_{\text{RHS}}^S + \mathbf{D}^S S^S I_z^S \cdot E^S x_2^S \]
\[ b^S = -\mathbf{D}^S S^S I_z^S \cdot E^S I_x^S \]
\[ c^S = -\mathbf{D}^S S^S I_z^S \cdot E^S I_d^S I_x^S \]  

(39)

\( a^S, b^S \) and \( c^S \) do not change during the iteration process (Newton Raphson). In order to calculate the aerodynamic forces on all load points, it is necessary to assemble the vectors \( H_{\text{RHS}}^S \text{step} \lambda \text{iter} n \) of all wing segments. In other words, the matrices \( a^S, b^S \) and \( c^S \) have to be assembled. This operation is immediate because the wing surfaces have no aerodynamic panels in common. Let \( a, b \) and \( c \) be called the matrices obtained after the assembling process. At global level, equation (38) is written as

\[ H_{\text{RHS}}^\text{step} \lambda \text{iter} n = a + bx^\text{step} \lambda \text{iter} n + cu^\text{step} \lambda \text{iter} n \]  

(40)

Using equation (9), the aerodynamic forces at load step \( \lambda \) and iteration \( n \) applied to the load points of the aerodynamic panels are:

\[ L^\text{step} \lambda \text{iter} n = \lambda L_{\text{ref}} I_\Gamma \left[ A^\Gamma \right]^{-1} (a + bx^\text{step} \lambda \text{iter} n + cu^\text{step} \lambda \text{iter} n) \]  

(41)

In a compact form, the previous equation can be written as

\[ L^\text{step} \lambda \text{iter} n = \lambda L_{\text{ref}} (\bar{a} + \bar{b}x^\text{step} \lambda \text{iter} n + \bar{c}u^\text{step} \lambda \text{iter} n) \]  

(42)

C. Aerodynamic Load Transferring

Relation (42) gives the aerodynamic loads applied to the load points located on the aerodynamic panels (first quarter). In order to use a FEM structural solver, the loads have to be transferred to the structural nodes using a good approximation. In the approximation of the shape using splines it was used assumed that the approximation between structural points and control points (in order to obtain the value of the displacements in the direction perpendicular to the reference plane) was constant. This is correct if the displacements are not too large, and it is consistent with the linear aerodynamic theory used. In other words, this assumption leads to the constant matrices \( \bar{a}, \bar{b} \) and \( \bar{c} \). A similar approach is adopted in the transferring of the aerodynamic forces from the load points to the structural nodes: the transformation is fixed during the iteration process. Therefore, the matrices used in this transformation will be constant and they can be calculated once.

Using the algorithm shown in Figure (9), it is possible to find the triangular structural element which contains the load point in which a concentrated load representing the lifting force is applied. Therefore, it is possible to determine what are the elements that contain the aerodynamic forces. This operation is done in the initial undeformed configuration. Consider the \( k \)th panel. The load point \( P_k \) is contained by the structural element \( m \). The equivalent nodal forces applied to the nodes of the element \( m \) are calculated using the area coordinates.

D. Assembly Process

Consider the \( k \)th panel. The aerodynamic loads on the load point relative to the panel \( k \) will be obtained by extracting from the vector \( L^\text{step} \lambda \text{iter} n \) the rows \( [3 \{ k - 1 \} + 1], [3 \{ k - 1 \} + 2] \) and \( [3 \{ k - 1 \} + 3] \) respectively.
This is equivalent to the definition of the matrices $\mathbf{a}^{*k}$, $\mathbf{b}^{*k}$ and $\mathbf{c}^{*k}$ obtained from the corresponding matrices $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ by extracting the same rows. Notice that the operation of extracting the rows is not performed on the vectors $\mathbf{x}^{\text{step} \lambda \text{iter} n}$ and $\mathbf{u}^{\text{step} \lambda \text{iter} n}$. Using the matrix $H^m$ which depends on the area coordinates of element $m$, the aerodynamic forces applied to the nodes of element $m$ can be written as

$$L^m_{\text{str} \lambda \text{iter} n} = \lambda L^m_{\text{ref}} h^m \left( \mathbf{a}^{*k} + \mathbf{b}^{*k} \mathbf{x}^{\text{step} \lambda \text{iter} n} + \mathbf{c}^{*k} \mathbf{u}^{\text{step} \lambda \text{iter} n} \right)$$

(43)

Carrying out the multiplications, assembling at structural level and adding some zero rows in correspondence of the rotational DOFs, the vector of the aerodynamic forces applied at the structural nodes is written as

$$L^{\text{step} \lambda \text{iter} n} = \lambda L_{\text{ref}} \left( \mathbf{A} + \mathbf{B} \mathbf{x}^{\text{step} \lambda \text{iter} n} + \mathbf{C} \mathbf{u}^{\text{step} \lambda \text{iter} n} \right)$$

(44)

V. Solution of the Nonlinear System Using the Newton Raphson Method

The wing is loaded by the aerodynamic loads and other loads such as the inertial loads. The used procedure (Newton Raphson) is the following:

- **Step # 1**

  The reference aerodynamic pressure $L_{\text{ref}}$ is calculated:

  $$L_{\text{ref}} = -\frac{\rho_{\infty} V^2_{\infty}}{N_{\text{step}}}$$

  (45)

  Similar operation can be done for the external concentrated load. The reference amplitude $P_{\text{ref}}$ of that loads will be

  $$P_{\text{ref}} = \frac{1}{N_{\text{step}}}$$

  (46)

- **Step # 2**

  The Newton Raphson procedure can be started. Consider now the first iteration in the generic load step $\lambda$. The aerodynamic loads (indicated here with $L^{\text{step} \lambda \text{iter} 1}_{\text{str}}$ in order to show that the aerodynamic tangent matrix is not included), are calculated using the following relation:

  $$L^{\text{step} \lambda \text{iter} 1}_{\text{str}} = \lambda \cdot L_{\text{ref}} \cdot \left( \mathbf{A} + \mathbf{B} \cdot \mathbf{x}^{\text{step} \lambda \text{iter} 1} \right)$$

  (47)
where \( \lambda \) is the load factor and it is equal to 1 for the first load step, 2 for the second load step and so forth. Notice that the vector \( \mathbf{x}^{\text{step} \lambda \text{iter} 1} \) is the vector \( \mathbf{x} \) calculated using the current coordinates of the structural nodes at the start of the iteration 1. The non-aerodynamic loads are calculated in a similar way:

\[
P^{\text{step} \lambda} = \lambda \cdot P_{\text{ref} \cdot P_{\text{ext}}}
\]

Notes that the non-aerodynamic loads are only load step dependent. At the first iteration of load step \( \lambda \), the following linear system is solved:

\[
\begin{align*}
\left( K^{\text{step} \lambda \text{iter} 1}_T - \lambda \cdot L_{\text{ref}} \cdot C \right) \cdot \mathbf{u}^{\text{step} \lambda \text{iter} 1} &= \left( L^{\text{step} \lambda \text{iter} 1}_\text{str} + P^{\text{step} \lambda}_\text{str} \right) - F^{\text{step} \lambda \text{iter} 1}_\text{int} = P^{\text{step} \lambda \text{iter} 1}_\text{unb} \\
\end{align*}
\]

\( K^{\text{step} \lambda \text{iter} 1}_T \) is the structural tangent matrix calculated at the first iteration of load step \( \lambda \), \( -\lambda \cdot L_{\text{ref}} \cdot C \) is the aerodynamic tangent matrix and \( P^{\text{step} \lambda \text{iter} 1}_\text{unb} \) is the unbalanced load. In the second iteration, the current aerodynamic external load is updated. Thus:

\[
L^{\text{step} \lambda \text{iter} 2}_\text{str} = \lambda \cdot L_{\text{ref}} \left( A + B \cdot \mathbf{x}^{\text{step} \lambda \text{iter} 2} \right)
\]

Notice that the vector \( \mathbf{x}^{\text{step} \lambda \text{iter} 2} \) is calculated at the beginning of the second iteration (of the load step \( \lambda \)). The structural tangent matrix and the coordinates of the nodes are updated and the following system is solved:

\[
\begin{align*}
\left( K^{\text{step} \lambda \text{iter} 2}_T - \lambda \cdot L_{\text{ref}} \cdot C \right) \cdot \mathbf{u}^{\text{step} \lambda \text{iter} 2} &= \left( L^{\text{step} \lambda \text{iter} 2}_\text{str} + P^{\text{step} \lambda}_\text{str} \right) - F^{\text{step} \lambda \text{iter} 2}_\text{int} = P^{\text{step} \lambda \text{iter} 2}_\text{unb}
\end{align*}
\]

The procedure is repeated until the desired tolerance is reached. Figure 7 shows the procedure described above.

Figure 7. Newton Raphson method with aerodynamic loads and external non-aerodynamic loads.
VI. Flutter Calculation

Given the freestream velocity \( V_\infty \), the aeroelastic deformation corresponding to the steady equilibrium is found using the methods described above. Then the system is linearized at that equilibrium configuration and a linear flutter computation is performed. This last operation is executed using an unsteady aerodynamic code based on Piecewise Continuous-Kernel Function Method (PCKFM) which uses as inputs the modes of the structure. It is important to underline that the modes must be calculated using the effective stiffness matrix which corresponds in this case to the structural tangent matrix calculated at the equilibrium state. The unsteady aerodynamic code calculates the generalized aerodynamic matrices for the assigned reduced frequencies (the reduced frequency definition is \( k = \omega b/V_\infty \), where \( b \) is a reference half chord). Once the generalized aerodynamic matrices are calculated, a fitting process is performed to transfer unsteady aerodynamic matrices from the frequency domain to the Laplace (and time) domains (Roger’s approximation). As a result of these operations, the following equation (the linearization about the steady solution is performed) can be written in the Laplace domain (no structural damping is considered in the equations here):

\[
\begin{bmatrix}
  s^2 \mathcal{M} + \mathcal{K}_T - \frac{1}{2} \rho_\infty V_\infty^2 \mathcal{A}(s)
\end{bmatrix} q(s) = 0
\]  

(52)

where \( \mathcal{M} \) is the generalized mass matrix, \( \mathcal{K}_T \) is the generalized stiffness matrix (obtained from the tangent stiffness matrix calculated at the steady equilibrium and the modes calculated at the steady equilibrium), \( q(s) \) is the vector representing the generalized coordinates and the generalized aerodynamic matrix \( \mathcal{A}(s) \) is known only in an approximated form (Roger approximation):

\[
\mathcal{A}(s) = \mathcal{A}_0 + \frac{s b}{V_\infty} \mathcal{A}_1 + \frac{s^2 b^2}{V_\infty^2} \mathcal{A}_2 + \frac{s}{s + \beta_1 \frac{b}{V_\infty}} \mathcal{A}_3 + \frac{s}{s + \beta_2 \frac{b}{V_\infty}} \mathcal{A}_4 + \ldots
\]  

(53)

where \( \frac{s}{s + \beta_1 \frac{b}{V_\infty}} \mathcal{A}_3, \frac{s}{s + \beta_2 \frac{b}{V_\infty}} \mathcal{A}_4 \) are the first two lag terms. For the sake of simplicity, consider the case in which only two lag terms are used (in the Joined Wing analysis 6 lag terms are used). Introducing the stiffness matrix of the coupled aerodynamic and structural systems \( \mathcal{K}^* = \mathcal{K}_T - \frac{1}{2} \rho_\infty V_\infty^2 \mathcal{A}_0 \), the mass matrix of the coupled aerodynamic and structural systems \( \mathcal{M}^* = \mathcal{M} - \frac{1}{2} \rho_\infty b^2 \mathcal{A}_0 \), the aerodynamic damping matrix \( \mathcal{C}^* = -\frac{1}{2} \rho_\infty b V_\infty \mathcal{A}_1 \), equation (52) can be rewritten (only two lag terms are considered) as

\[
\begin{bmatrix}
  s^2 \mathcal{M}^* + s \mathcal{C}^* + \mathcal{K}^* - \frac{1}{2} \rho_\infty V_\infty^2 \frac{s}{s + \beta_1 \frac{b}{V_\infty}} \mathcal{A}_3 - \frac{1}{2} \rho_\infty V_\infty^2 \frac{s}{s + \beta_2 \frac{b}{V_\infty}} \mathcal{A}_4
\end{bmatrix} q(s) = 0
\]  

(54)

or, in a compact form:

\[
\begin{bmatrix}
  s^2 \mathcal{M}^* + s \mathcal{C}^* + \mathcal{K}^* - \frac{s}{s + \beta_1} \mathcal{A}_3 - \frac{s}{s + \beta_2} \mathcal{A}_4
\end{bmatrix} q(s) = 0
\]  

(55)

Introducing the state variables

\[
\begin{align*}
X_1 &= q \\
X_2 &= s q \Rightarrow X_2 = s X_1 \\
X_3 &= \frac{q}{s + \beta_1} \Rightarrow X_3 = -\beta_1 X_3 + X_2 \\
X_4 &= \frac{q}{s + \beta_2} \Rightarrow X_4 = -\beta_2 X_4 + X_2
\end{align*}
\]  

(56)

equation (55) and (56) can be combined as

\[
\begin{bmatrix}
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathcal{M}^* & 0 & 0 & 0 & 0 & -\mathcal{K}^* & -\mathcal{C}^* & q \mathcal{A}_3 \\
0 & 0 & I & 0 & 0 & -\beta_1 I & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & -\beta_2 I & 0 & 0
\end{bmatrix} \begin{bmatrix}
X_1 \\
X_2 \\
X_3 \\
X_4
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
-\mathcal{K}^* & -\mathcal{C}^* & q \mathcal{A}_3 & q \mathcal{A}_4 \\
0 & I & -\beta_1 I & 0 \\
0 & I & 0 & -\beta_2 I
\end{bmatrix} \begin{bmatrix}
X_1 \\
X_2 \\
X_3 \\
X_4
\end{bmatrix}
\]  

(57)
or

\[ U \cdot sX = V \cdot X \]  \hspace{1cm} (58)

Where \( X \) is the state vector with displacements, speeds and aerodynamic states. The eigenvalues (in general complex) of the system of equations are in the form

\[ \lambda_j = \sigma_j + i\omega_j \]  \hspace{1cm} (59)

And they give a measure of the stability of the system (if \( \sigma_j > 0 \) the system is unstable). Given a freestream velocity \( V_{\infty} \), it is possible to calculate the physical eigenvalues considering that the Roger fitting process is performed up to a certain reduced frequency \( k_{\text{max}} \):

\[ \omega_{\text{max}} = k_{\text{max}} V_{\infty} / b \]  \hspace{1cm} (60)

If an eigenvalue has \( \omega_j > \omega_{\text{max}} \) then it is discarded. For a range of speeds \( V_{\infty} \) the steady equilibrium is calculated, then the system is linearized as explained above and the eigenvalues are calculated and, therefore, the root locus can be plotted giving information on the “stability in the small” of the system. The procedure is summarized in Figure 8.

![Figure 8. Nonlinear flutter calculation. Description of the used procedure](image)

VII. Results

The present procedure has been tested for different problems and comparison with other available results present in the literature was performed.

A. Aerodynamic Validation

The aerodynamic vortex lattice formulation has been tested for several different planar configurations: wings with no sweep angle, swept-forward wings and swept-back wings. In all examined cases the correlation with other results present in the literature was excellent.
B. Static Aeroelastic Validation

The static aeroelastic validation is performed for a linear and nonlinear two-dimensional cases (the comparison is made with an analytical solution) and a nonlinear case represented by a delta wing.

1. Linear Two-Dimensional Case

This test case reproduces a two-dimensional case for which an analytical solution is available. The two-dimensional case is analyzed using a high-aspect-ratio rectangular wing (see Figure 9). The material properties are chosen in a way to have the plate elements much stiffer than the stiffness of the springs. With this method, the wing behaves like a rigid wing with a pitch spring (see Figure 10). In order to approximate the

\[
\begin{align*}
1 &= (+a, -10a, 0) \\
2 &= (0, -10a, +a\tan\alpha) \\
3 &= (0, 0, +a\tan\alpha) \\
4 &= (+a, 0, 0)
\end{align*}
\]

The equations of motion for the wing can be written as:

\[
\begin{align*}
&V_\infty \\
&\text{Line of the first quarter} \\
&\text{Elastic Axis}
\end{align*}
\]

\[
\begin{align*}
a &= 1 \text{ mm} & \frac{a}{h} &= 100 & h &= 10^{-2} \text{ mm} \\
\nu &= 0.45 & E &= 3.3 \times 10^{15} \frac{\text{Kg}}{\text{mm}^2} \\
\rho_\infty &= 1.225 \times 10^{-3} \frac{\text{Kg}}{\text{m}^3} & S &= 10a^2 \alpha = \frac{\alpha}{180}
\end{align*}
\]

2D case, two springs with stiffness \( K_\varphi \) are positioned in \( P_1 \) and \( P_2 \) (see Figure 9). All nodes of the wing are free to move except points \( P_1 \) and \( P_2 \): the displacements and the rotations along \( x \) and \( z \) are constrained to

\[
\begin{align*}
qSC_L &= qSC_{L_\infty}(\alpha + \theta) \\
M_\varphi &= K_\varphi \theta
\end{align*}
\]

Figure 9. Geometry of the wing.

Figure 10. Equilibrium of the airfoil.
be zero and the rotation along \( y \) is allowed (pitching DOF). However, in order to correctly simulate the 2D wing the stiffness matrix of the code is corrected adding in the DOFs corresponding to the points \( P_1 \) and \( P_2 \) (pitching DOF) the stiffness \( K_\vartheta \). Considering Figure 10, it is possible to find the divergence speed and the elastic deformation \( \vartheta \) of the wing. The divergence speed can be calculated from the relation

\[(0.55a q_{\text{div}} SC_{L_\alpha} - K_\vartheta) \vartheta = 0 \Rightarrow q_{\text{div}} = \frac{K_\vartheta}{0.55a SC_{L_\alpha}} = \frac{K_\vartheta}{5.5a^3 C_{L_\alpha}} \quad (61)\]

Using the wing shown in figure 9 (notice that no symmetry condition is imposed), it is possible to calculate the quantity \( C_{L_\alpha} \). For this wing \( C_{L_\alpha} \) was found to be 4.8979. Assuming \( K_\vartheta = 200 \frac{Kg}{mm^2 s^2} \), the divergence dynamic pressure is:

\[ q_{\text{div}} = 200 \cdot 1.3 \cdot 4.8979 = 7.35 \frac{Kg}{mm s^2} \quad (62) \]

The value found using the present capability (via eigenvalue analysis) is

\[ q_{\text{div}} = 7.35 \frac{Kg}{mm s^2} \quad (63) \]

Consider again Figure 10. The elastic rotation \( \vartheta \) can be found using the equilibrium of the moments:

\[ qSC_{L_\alpha} (\alpha + \vartheta) 0.55a = K_\vartheta \vartheta \Rightarrow \vartheta = \frac{qSC_{L_\alpha} \alpha 0.55a}{K_\vartheta - qSC_{L_\alpha} 0.55a} \quad (64) \]

Remembering that for this case \( S = 10a^2 = 10 \), \( C_{L_\alpha} = 4.8979 \), \( a = 1 \) and \( \alpha = \frac{\pi}{180} \) it is possible to write:

\[ \vartheta = 0.4701646479 \frac{q}{200.0/26.93845q} \quad (65) \]

The values of the elastic rotation are reported in Table 1 and the comparison between the analytical value and the numerical value is made. The same wing positioned in the plane \( x - z \) (instead of the plane \( x - y \) as shown in Figure 9) was also studied and no difference of the results has been found. This last finding shows that the transformations in space (described above in the paper) are correct.

2. Nonlinear Two-Dimensional Case

In the nonlinear case the considered wing and material are the same (see Figure 9). The only difference is that the spring is assumed nonlinear. Imposing the equilibrium, it is possible to write:

\[ qSC_{L_\alpha} (\alpha + \vartheta) 0.55a = K_1 \vartheta + K_2 \vartheta^2 \quad (66) \]

Solving with respect to \( \vartheta \):

\[ \vartheta = \frac{-(K_1 - qSC_{L_\alpha} 0.55a) + \sqrt{(K_1 - qSC_{L_\alpha} 0.55a)^2 + 4K_2 qSC_{L_\alpha} \alpha 0.55a}}{2K_2} \quad (67) \]

<table>
<thead>
<tr>
<th>( q )</th>
<th>Analytical</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.012</td>
<td>0.012</td>
</tr>
<tr>
<td>4</td>
<td>0.020</td>
<td>0.021</td>
</tr>
<tr>
<td>5</td>
<td>0.036</td>
<td>0.037</td>
</tr>
</tbody>
</table>

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Assuming $K_1 = 100 \frac{Kg \cdot mm^2}{s^2}$ and $K_2 = 50 \frac{Kg \cdot mm^2}{s^2}$ (recall that these values are obtained using two springs in $P_1$ and $P_2$ so each spring has $K_1/2$ and $K_2/2$) it is possible to write:

$$\vartheta = -1.0 + 0.2693845q + 0.01\sqrt{(100.0 - 26.93845q)^2 + 94.03292958q^2}$$  \hspace{1cm} (68)$$

The values of the elastic rotation are reported in Table 2 and the comparison between the analytical value and the numerical value obtained with plate elements, vortex lattice mesh, and a Newton-Raphson iteration is made. Notice that in the nonlinear case the contribution to the structural tangent matrix related to the

<table>
<thead>
<tr>
<th>$q$</th>
<th>Analytical</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$6.41 \cdot 10^{-3}$</td>
<td>$6.50 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>2</td>
<td>$2.00 \cdot 10^{-2}$</td>
<td>$2.03 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>3</td>
<td>$6.31 \cdot 10^{-2}$</td>
<td>$6.48 \cdot 10^{-2}$</td>
</tr>
</tbody>
</table>

spring in $P_1$ is $K_T = K_1/2+2(K_2/2) \cdot (\vartheta)$ and similarly for the spring positioned in $P_2$. At each iteration, also, the initial elastic moment of the springs have to be taken into account in the calculation of the unbalanced loads.

3. Delta Wing

The geometry (see Ref. 30) of the wing is shown in figure 12. The present capability has been obtained considering trapezoidal wing segments. For that reason, the delta wing is analyzed using the model shown in the right part of Figure 12. The comparison with some reference results (courtesy by Peter J. Attar) is made in Table 3. In the table normalized tip deflections $w/h$ are shown for different root angles of attacks and flight speeds.

C. Validation of the Nonlinear Flutter Calculation Procedure

The present procedure has been validated comparing current flutter calculation with some linear\textsuperscript{31} and nonlinear\textsuperscript{30} flutter results available in the literature. The linear flutter results were compared for a forward-swept wing with sweep angle of $-\frac{\pi}{6}$ and a double-swept wing with inboard sweep angle of $+\frac{\pi}{6}$ and outboard
Figure 12. Geometry of the Analyzed Wing.

![Diagram of Analyzed Wing](image)

- $a = 381\text{mm}$
- $h = 238\text{mm}$
- $t = 1.6\text{mm}$
- $\nu = 0.45$
- $E = 3.3 \times 10^3 \text{N/mm}^2$
- $\rho_\infty = 1.225 \times 10^{-9} \text{kg/mm}^3$

D. Aeroelasticity of a Joined Wing Configuration

1. Structural Comparison with NASTRAN

The geometry of the wing is shown in figure [13]. The comparison with NASTRAN is shown if Figure [14] (in figure 14 a mesh with 620 triangular elements and 377 nodes was used. In the other cases the meshes are slightly different). The NASTRAN validated is for a structural case only with a uniform pressure load on both wings, which is increased gradually. As is well known in the case of joined-wing configurations aeroelastic behavior is highly sensitive to the type of joint between the main and rear wings and its location. The joint used in the present study is just one out of many possibilities. The thrust of the work here is to demonstrate analysis and simulation tools for the joined-wing case. Systematic studies of joint-types is planned for the near future. Also note that the cases analyzed in this paper cover a cantilevered joined-wing only. Extension to other boundary conditions along the fuselage or free-free flight is straightforward.
Table 3. Tip displacement. Comparison with the reference results (courtesy by Peter J. Attar).

<table>
<thead>
<tr>
<th>$V_\infty [m/s]$</th>
<th>Reference $w/h$</th>
<th>Present $w/h$</th>
<th>Reference $w/h$</th>
<th>Present $w/h$</th>
</tr>
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<tbody>
<tr>
<td>21</td>
<td>8.80</td>
<td>8.07</td>
<td>17.08</td>
<td>15.51</td>
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<tr>
<td>23</td>
<td>9.94</td>
<td>9.12</td>
<td>19.25</td>
<td>17.50</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$V_\infty [m/s]$</th>
<th>Reference $w/h$</th>
<th>Present $w/h$</th>
<th>Reference $w/h$</th>
<th>Present $w/h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>21</td>
<td>24.98</td>
<td>22.53</td>
<td>32.66</td>
<td>29.25</td>
</tr>
<tr>
<td>23</td>
<td>28.15</td>
<td>25.41</td>
<td>36.80</td>
<td>32.97</td>
</tr>
</tbody>
</table>

Table 4. Flutter velocity and frequency ($\alpha = \frac{\pi}{180}$). Comparison with the reference results (experimental).

<table>
<thead>
<tr>
<th>$V_\infty [m/s]$</th>
<th>$f [Hz]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>24.5</td>
<td>14.5</td>
</tr>
<tr>
<td>25.1</td>
<td>15.0</td>
</tr>
</tbody>
</table>
2. Nonlinear Divergence

In order to put the aeroelastic results in a non-dimensional form, the reference aerodynamic speed is defined as

\[ V_{\text{ref}} = \sqrt{\frac{Eh^4}{\rho_\infty a^4}} \]  \hspace{1cm} (69)

In the analyzed model \( E = 69 \times 10^9 \text{N/m}^2; \rho_\infty = 1.225 \text{Kg/m}^3; a = 0.2 \text{m}; c = 2 \times 10^{-3} \text{m}. \) Therefore, the reference speed is:

\[ V_{\text{ref}} = \sqrt{\frac{69 \times 10^9 \times (2 \times 10^{-3})^4}{1.225 \times 0.2^4}} \simeq 23.7332 \text{ m/s} \]  \hspace{1cm} (70)

It is possible to define another reference speed. Consider the swept forward wing of the joined wing. Suppose that the wing is on the plane \( x - y \) (it is not true in the analyzed configuration because of the dihedral angle, but in the definition of a reference speed this is not relevant). Using the geometry of the wing system (see Figure 13), it can be deduced that the negative sweep angle for the swept forward wing is:

\[ \Lambda = -\frac{\pi}{4} \]  \hspace{1cm} (71)

Defining the parameter \( a_0 \) is in general different than \( 2\pi \). The goal is to define a reference speed, therefore the assumption \( a_0 = 2\pi \) can be accepted even without any correction.

The parameter \( a_0 \) is in general different than \( 2\pi \). The goal is to define a reference speed, therefore the assumption \( a_0 = 2\pi \) can be accepted even without any correction.
All following results will be put in a non-dimensional form using $V_{\text{ref}}$. However, the results can be expressed using the reference speed $V_{\text{ref}}^*$ and observing that $V_{\text{ref}}/V_{\text{ref}}^* = 3.24471$.

The nonlinear divergence is calculated evaluating the gradient of the displacements: when the gradient is very large, the divergence is assumed reached. The nonlinear deformation behavior with increasing air speed for different cases of initial root angles of attack is shown in Figures 15 and 16. Consider now a typical deformed shape of the joined wing (Figure 17). The angles of attack are evaluated for a certain speed at the positions shown in Figure 18. Therefore, it is possible to calculate the twist distribution under aerodynamic loads (see also Table 5). The initial angle of attack is constant along the span in the initial configuration. The twist distributions are plotted for the cases $\alpha = \pi/180$, $\alpha = 4\pi/180$ and $\alpha = 8\pi/180$ in Figure 18. From Figure 18 it can be learned that the swept-back wing shows a reduction of the angle of attack (stable behavior), while the swept-forward wing has an opposite behavior. It is interesting to examine the area near the joint: the swept-back wing opposes the unstable behavior of the swept-forward wing. The swept-back wing, thus, stabilizes the divergence tendencies of the swept-forward wing.

3. **Effect on Aerodynamic Loads on the Frequencies**

It interesting to analyze how the frequencies (free vibration problem) change when the wing system is aeroelastically loaded due to a freestream velocity. The cantilevered case with $\alpha = 4\pi/180$ is considered first with no freestream speed (free vibration case without load) then with a speed corresponding to the reference speed. The comparison with NASTRAN is also made (for the case without aerodynamic loads only). As can be seen, the change of the frequencies with the aerodynamic loads depends on the geometry of the Joined Wing. Patil found, for example, that in a particular planar Joined Wing configuration the first frequencies increase when the load is applied, while in the corresponding non-planar Joined Wing configuration the behavior is opposite.
Figure 15. Non-dimensional aeroelastic displacement.

Figure 16. Non-dimensional aeroelastic displacement.
Figure 17. Definition of the local angles of attack.

Table 5. Cantilevered joined-wing. Twist distributions (in all cases $\frac{V_{\infty}}{V_{ref}} = 0.74$).

<table>
<thead>
<tr>
<th>$\frac{180\alpha_{01}}{\pi}$</th>
<th>$\frac{180\alpha_{02}}{\pi}$</th>
<th>$\frac{180\alpha_{03}}{\pi}$</th>
<th>$\frac{180\alpha_{04}}{\pi}$</th>
<th>$\frac{180\alpha_{05}}{\pi}$</th>
<th>$\frac{180\alpha_{06}}{\pi}$</th>
<th>$\frac{180\alpha_{07}}{\pi}$</th>
<th>$\frac{180\alpha_{08}}{\pi}$</th>
<th>$\frac{180\alpha_{09}}{\pi}$</th>
<th>$\frac{180\alpha_{10}}{\pi}$</th>
<th>$\frac{180\alpha_{11}}{\pi}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00</td>
<td>0.70</td>
<td>0.69</td>
<td>0.86</td>
<td>1.03</td>
<td>1.04</td>
<td>1.29</td>
<td>1.55</td>
<td>1.48</td>
<td>1.00</td>
<td></td>
</tr>
<tr>
<td>2.00</td>
<td>1.38</td>
<td>1.42</td>
<td>1.84</td>
<td>2.13</td>
<td>2.11</td>
<td>2.07</td>
<td>2.62</td>
<td>3.22</td>
<td>3.04</td>
<td>2.00</td>
</tr>
<tr>
<td>3.00</td>
<td>2.01</td>
<td>2.19</td>
<td>2.92</td>
<td>3.32</td>
<td>3.20</td>
<td>3.10</td>
<td>3.97</td>
<td>4.93</td>
<td>4.64</td>
<td>3.00</td>
</tr>
<tr>
<td>4.00</td>
<td>2.58</td>
<td>2.95</td>
<td>4.06</td>
<td>4.55</td>
<td>4.33</td>
<td>4.15</td>
<td>5.35</td>
<td>6.71</td>
<td>6.28</td>
<td>4.00</td>
</tr>
<tr>
<td>5.00</td>
<td>3.11</td>
<td>3.67</td>
<td>5.20</td>
<td>5.81</td>
<td>5.48</td>
<td>5.22</td>
<td>6.79</td>
<td>8.57</td>
<td>7.97</td>
<td>5.00</td>
</tr>
<tr>
<td>6.00</td>
<td>3.61</td>
<td>4.36</td>
<td>6.32</td>
<td>7.06</td>
<td>6.66</td>
<td>6.31</td>
<td>8.26</td>
<td>10.45</td>
<td>9.68</td>
<td>6.00</td>
</tr>
<tr>
<td>7.00</td>
<td>4.11</td>
<td>5.02</td>
<td>7.41</td>
<td>8.30</td>
<td>7.85</td>
<td>7.41</td>
<td>9.73</td>
<td>12.34</td>
<td>11.38</td>
<td>7.00</td>
</tr>
<tr>
<td>8.00</td>
<td>4.62</td>
<td>5.67</td>
<td>8.47</td>
<td>9.52</td>
<td>9.05</td>
<td>8.52</td>
<td>11.19</td>
<td>14.20</td>
<td>13.06</td>
<td>8.00</td>
</tr>
</tbody>
</table>
Figure 18. Twist distribution.
Figure 19. Mode 1 with and without aerodynamic loads.
Figure 20. Mode 2 with and without aerodynamic loads.
Figure 21. Mode 3 with and without aerodynamic loads.
Figure 22. Mode 4 with and without aerodynamic loads.

\[
\omega \sqrt{\frac{I_x}{I_B}} = 382.66 \cdot 10^{-3}
\]

\[
\frac{V_\infty}{V_{ref}} = 0
\]

Present
\[
\omega \sqrt{\frac{I_x}{I_B}} = 382.94 \cdot 10^{-3}
\]

Present
\[
\frac{V_\infty}{V_{ref}} = 1
\]

\[
\omega \sqrt{\frac{I_x}{I_B}} = 379.14 \cdot 10^{-3}
\]

Mode 4
NASTRAN

\[ \omega \sqrt{\frac{EL}{K^2}} = 479.77 \times 10^{-3} \]

\[ \frac{V_0}{V_{ref}} = 0 \]

Present

\[ \omega \sqrt{\frac{EL}{K^2}} = 479.86 \times 10^{-3} \]

Present

\[ \frac{V_0}{V_{ref}} = 1 \]

\[ \omega \sqrt{\frac{EL}{K^2}} = 466.61 \times 10^{-3} \]

Figure 23. Mode 5 with and without aerodynamic loads.
4. Nonlinear Flutter Calculation: Root Locus

Using the procedure described above, the root locus for dynamic stability-in-the-small (linearized flutter about states of nonlinear static equilibrium) has been created for the nonlinear case and the linear case ($\alpha = \frac{4\pi}{180}$). In figure 24 the root locus, for the nonlinear case, is plotted. Speeds between $V_\infty = 15 \text{ m/s}$ and $V_\infty = 21 \text{ m/s}$ have been considered. It is important to observe in Figures 15 and 16 that for the case $\alpha = \frac{4\pi}{180}$ the speed for which the deformation starts growing considerably (nonlinear divergence) can be estimated to be $\frac{V_\infty}{V_{Ref}} = 0.8 \div 0.9$. This implies that, for the examined case, the speed $V_\infty = 21 \text{ m/s}$ is near to the interval of speeds in which the deformation grows considerably. In figure 24 it can be seen that for most of the modes the branches becomes more stable when the speed is increased. For the low frequency modes, there is an exception: mode 4 becomes less stable when the speed is increased.

It is interesting to see the effects of the geometrical nonlinearity on the root locus. In Figures 25-29 it can be observed that the behavior in the nonlinear case (in which at each speed the modes are recalculated using the procedure described in Figure 8) is very different. The reason is that in the linear case the structural stiffness matrix does not change and the modes do not change when the speed is increased. The root loci shown contains points for high speeds, such $30 \text{ m/s}$. Above $21 \text{ m/s}$ aeroelastic poles of the linearized system begin to behave in a complex way. At such speed the wing approaches divergence, deformations (and twists) are large. For example, the local angle of attack in different positions (see Figure 17) has been calculated. For example, for the speed $V_\infty = 30 \text{ m/s}$ it was found $\alpha_{09} = 21.91 \frac{\pi}{180}$ raising concerns about the validity of linear aerodynamic theory. Stability at high speeds with significant geometric-stiffness structural nonlinearity has to be investigated thoroughly. A time domain transient simulation capability based on the nonlinear structural equations of motion and time domain aerodynamics, already in place, will be used for that purpose, and results will be reported in subsequent publications.
Figure 25. Root locus. Linear and nonlinear cases.

Figure 26. Root locus. Linear and nonlinear cases.
Figure 27. Root locus. Linear and nonlinear cases.

Mode 3

Figure 28. Root locus. Linear and nonlinear cases.

Mode 4

Linear

Nonlinear
VIII. Conclusions

A capability for divergence and dynamic stability-in-the-small analysis for joined-wings configuration with geometric structural nonlinearities has been described. A nonlinear static aeroelastic iterative process for finding equilibrium positions depending on initial angles of attack of cantilevered wings and on flight speeds is augmented by a linearized time-domain flutter analysis about these equilibrium conditions. Accuracy of the new capability is validated by comparison to analytical solutions for rigid high aspect ratio wings on nonlinear pitch springs and a cantilvered nonlinear delta wing case. A simple joined-wing test case is used for exploratory studies. No studies of root boundary conditions (or free-free flight), joint-type and location effect, or structural mass distribution effects are reported. These, and time domain simulations, will be reported in upcoming publications.
References

