A Reduced Order Nonlinear Aeroelastic Analysis of Joined Wings Based on the Proper Orthogonal Decomposition

Luciano Demasi,* Antonio Palacios,†

A method for the reduced-order nonlinear aeroelastic analyses is presented. The structure is assumed to have important geometric nonlinearity whereas the aerodynamics is assumed to be linear. This is a typical situation of airplane configurations in which geometric stiffness effects are important but deformations are moderate, flow is attached and linear aerodynamic modeling is adequate. Joined-wings, strut-braced, and low aspect ratio wings are possible applications. The approach proposed in this paper is to reduce the order of the system by adopting a basis of structural tangent modes reconstructed via Proper Orthogonal Decomposition (POD) as a function of the aerodynamic pressure. It is shown that the proposed procedure provides excellent results in the range of aerodynamic speeds used to build the basis of tangent modes. Extrapolation of the functions used to interpolate the POD coefficients and generate the modal basis has been proven to provide satisfactory results for planar wings. The extrapolation is more challenging when Joined Wings are considered.

Nomenclature

\[\begin{align*}
\alpha & \quad \text{Angle of attack} \\
\sigma & \quad \text{Eigenvalues associated with POD modes} \\
\lambda & \quad \text{Integer representing a generic load step} \\
\lambda_{\text{max}} & \quad \text{Integer representing the maximum extrapolated value for the load step} \\
\zeta & \quad \text{Real number corresponding to a generic load step. } -1 \leq \zeta \leq +1 \\
V_\infty & \quad \text{Freestream velocity} \\
V & \quad \text{Generic velocity incrementally applied (Newton Raphson iteration)} \\
V_{\text{POD}} & \quad \text{Maximum velocity at which the POD analysis is performed without extrapolations} \\
\rho_\infty & \quad \text{Air density} \\
N_{\text{step}} & \quad \text{Number of load steps. The corresponding speed is } V_\infty \\
N_s & \quad \text{Number of data sets for a POD decomposition} \\
P_{\text{ref}} & \quad \text{Dimension-less number representing the reference aerodynamic load} \\
L_l & \quad \text{Legendre polynomial of order } l \\
u_1, u_2, u_3 & \quad \text{Translational Finite Element nodal displacements along } x, y \text{ and } z \text{ directions} \\
u_4, u_5, u_6 & \quad \text{Finite Element nodal rotations along } x, y \text{ and } z \text{ directions} \\
u_j & \quad \text{Generic Finite Element displacement or rotation} \\
\mathcal{N} & \quad \text{Number of low-frequency structural tangent modes used to build the basis} \\
\mathcal{M} & \quad \text{Number of POD modes used to reconstruct a submode} \\
u_j \phi_i & \quad s^\text{th} \text{ POD mode used to reconstruct a submode relative to } u_j \text{ and the } i^\text{th} \text{ structural tangent mode} \\
u_j a_i & \quad s^\text{th} \text{ POD coefficient used to reconstruct a submode relative to } u_j \text{ and the } i^\text{th} \text{ structural tangent mode} \\
L & \quad \text{Number of Legendre polynomials used in the fitting process (extrapolation)}
\end{align*}\]

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<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
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<tr>
<td>$u_j A_i^s$</td>
<td>$t^{th}$ coefficient of Legendre polynomial used to reconstruct the $s^{th}$ POD coefficient for a submode relative to $u_j$ and the $i^{th}$ structural tangent mode</td>
</tr>
<tr>
<td>$E$</td>
<td>Elastic modulus, total energy captured in a proper orthogonal decomposition</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Material density</td>
</tr>
<tr>
<td>$\nu$</td>
<td>Poisson's ratio</td>
</tr>
<tr>
<td>$\varepsilon_m$</td>
<td>Least square truncation error in POD reconstruction with $m$ modes</td>
</tr>
<tr>
<td>$v$</td>
<td>Eigenvectors of correlation matrix $C$</td>
</tr>
<tr>
<td>$v_i^k$</td>
<td>$i^{th}$ component of $k^{th}$ eigenvector of correlation matrix $C$</td>
</tr>
<tr>
<td>$h$</td>
<td>Thickness of the plate</td>
</tr>
<tr>
<td>$x, y, z$</td>
<td>Coordinate system</td>
</tr>
<tr>
<td>$x; y; z$</td>
<td>Coordinate system of spatio-temporal data for POD analysis</td>
</tr>
<tr>
<td>$t$</td>
<td>Continuous time</td>
</tr>
<tr>
<td>$t_i$</td>
<td>Discrete time</td>
</tr>
<tr>
<td>$u_x$</td>
<td>Displacement in the $x$ direction</td>
</tr>
<tr>
<td>$u_y$</td>
<td>Displacement in the $y$ direction</td>
</tr>
<tr>
<td>$u_z$</td>
<td>Displacement in the $z$ direction</td>
</tr>
<tr>
<td>$\Psi$</td>
<td>Matrix containing the shape vectors (basis)</td>
</tr>
<tr>
<td>$\Phi^i$</td>
<td>$i^{th}$ structural tangent mode</td>
</tr>
<tr>
<td>$\phi_s^i$</td>
<td>$s^{th}$ POD mode</td>
</tr>
<tr>
<td>$\Phi_j^i$</td>
<td>Generic structural tangent submode</td>
</tr>
<tr>
<td>$q$</td>
<td>Vector containing the generalized coordinates</td>
</tr>
<tr>
<td>$U$</td>
<td>Cumulative displacement vector</td>
</tr>
<tr>
<td>$u$</td>
<td>Displacement vector referred to the coordinates at the beginning of the current iteration</td>
</tr>
<tr>
<td>$f$</td>
<td>Scalar function with time- and space-dependent variables</td>
</tr>
<tr>
<td>$f_i$</td>
<td>$i^{th}$ component of spatio-temporal vector</td>
</tr>
<tr>
<td>$C$</td>
<td>Correlation matrix</td>
</tr>
<tr>
<td>$M$</td>
<td>Mass matrix</td>
</tr>
<tr>
<td>$S$</td>
<td>Tensor product matrix</td>
</tr>
<tr>
<td>$x^{pert}$</td>
<td>Vector containing the coordinates of the nodes of the perturbed configuration</td>
</tr>
<tr>
<td>$x^{pert}$</td>
<td>Augmented vector obtained from $x^{pert}$</td>
</tr>
<tr>
<td>$x^{\alpha=0}$</td>
<td>Vector containing the coordinates of the nodes of the reference configuration</td>
</tr>
<tr>
<td>$x^{\alpha=0}$</td>
<td>Augmented vector obtained from $x^{\alpha=0}$</td>
</tr>
<tr>
<td>$F_{\text{int}}$</td>
<td>Vector containing the internal forces</td>
</tr>
<tr>
<td>$P_{\text{ext}}$</td>
<td>Vector containing the non-aerodynamic forces</td>
</tr>
<tr>
<td>$K_T$</td>
<td>Structural tangent matrix</td>
</tr>
<tr>
<td>$K_{\text{aero}}$</td>
<td>Aerodynamic tangent matrix</td>
</tr>
<tr>
<td>$K_{\text{Tangent}}$</td>
<td>Aeroelastic tangent matrix</td>
</tr>
<tr>
<td>$P_{\text{unb}}$</td>
<td>Vector of unbalanced loads</td>
</tr>
<tr>
<td>$u_j a_i^s$</td>
<td>Vector containing the POD coefficients (as function of $\lambda$) relative to the $s^{th}$ POD mode used to reconstruct submode $u_j \Phi^i$</td>
</tr>
<tr>
<td>$L$</td>
<td>Rectangular matrix containing the Legendre polynomials</td>
</tr>
<tr>
<td>$u_j A_i^s$</td>
<td>Vector containing the coefficients of the Legendre polynomials</td>
</tr>
<tr>
<td>$u_j a_i^s$</td>
<td>Vector which contains the absolute values of the coefficients relative to the $s^{th}$ POD mode used to reconstruct the structural tangent submode $u_j \Phi^i$</td>
</tr>
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</table>

### I. Introduction

In many aeroelastic cases linearized aerodynamics based on potential methods or even linearized CFD methods is completely adequate, and the computational cost compared to fully detailed CFD solutions much smaller. However, very often the structural simulation needs to take into account the geometric nonlinearity even if the aerodynamics can be assumed linear. The simulation of geometrical nonlinearity requires the computationally expensive solution of many large linear systems (Newton Raphson method).
The reduction of the number of structural degrees of freedom of a given model for computational gains and for coupling with modal-based generalized unsteady aerodynamics is, thus, of significant importance. Modal order reduction procedures for geometrically nonlinear structures were introduced based on mode shapes of the structure. Reducing the order of both the structural and aerodynamic models has been quite successful in linear aeroelasticity: it is common practice to calculate the “classical flutter” speed with a set of generalized matrices obtained from a small number of global shape modes. In the nonlinear case the structural modes, eigenvectors of the free vibration problem, change with the load level due to the fact that the tangent stiffness matrix changes (geometric effect). This complicates the problem of modal reduction: a constant set of modes is inadequate to simulate the nonlinear geometric effects and the error of the approximation is already unacceptable after a small number of load steps. These effects are directly related to the degree of nonlinearity, which is a problem-dependent property.

Some non-conventional airplane configurations such as Joined Wings (JW) and Truss Braced Wings (TBW) present significant in-plane forces in correspondence of the joint areas. This implies a quite strong nonlinear geometric effect even for small angle of attack and flow attached. The aerodynamics can still be assumed linear whereas the structural effects need to be simulated with the inclusion of geometric nonlinear effects. A reduction of the order is very appealing for two main reasons. The first reason is that the computational cost is greatly improved when the number of degrees of freedom is reduced. The second reason is that classical aeroelastic computational tools used by the industry today are based on a modally reduced aerodynamic approaches in the frequency domain.

A basis built by adopting structural tangent modes (which are calculated from the solution of a free vibration problem with the actual tangent stiffness matrix) is often enriched with the use of higher order modes. The basic idea is to have a basis which takes into account the variability of the modes due to the nonlinear effects. The results can be improved if the basis is updated during the iterative procedure which is required to solve the static or dynamic problem. One possibility of numerical improvements is to use the “stress modes” stress distributions obtained from the full order mathematical model for some assigned modal deformation shapes. New efforts and studies of modal reduction techniques has been recently performed by several authors. A recent approach is based on the following main ideas. First, the equations governing the problem are written in terms of generalized coordinates by using a set of modes. This can be generalized to the case of thermal problems and functionally graded materials. Second, the reduced order nonlinear stiffness matrix is written as quadratic and cubic functions of the modal coordinates. These functions contain several coefficients representing the quadratic and cubic modal stiffnesses which are a priori not known. Third, the quadratic and cubic modal stiffnesses are determined from a series of static nonlinear finite element solutions as proposed by Muravyov and Rizzi. In particular, a series of specified displacement fields in physical coordinates are assigned. The magnitude of the the prescribed displacement field has to be physically meaningful. The number of required nonlinear static solutions depends on the number of modes selected to build the basis. For example, it can be shown that 12 modes require the solution of 454 nonlinear static problems. The evaluation of the quadratic and cubic modal stiffness coefficients is performed independent of loading. This implies that the procedure needs only be performed once prior to the solution of the reduced order set of differential equations.

Alternative approaches to deal with aeroelastic problems in which the geometrically nonlinear effects are the dominant physical aspects of the problem (JW and TBW) have been recently proposed. The idea behind these approaches is to reduce the order of the aerodynamic model (classical modally-reduced aerodynamics based on the concept of generalized aerodynamic force matrix) but keep the geometrically nonlinear structural model at full order level. That is: the structural model is not reduced in order and, therefore, the numerical difficulties inherent with the modal reduction of nonlinear structural problems are avoided and the aeroelastic mathematical representation can be quite accurate in the description of local effects and instabilities (buckling and/or divergence). This is accomplished with an energetic approach and the Least Square Method (LSM) technique. Another advantage is in the possibility of use existing well established and reliable nonlinear structural codes such as NASTRAN, ANSYS etc. and linear unsteady modally reduced aerodynamics for new accurate descriptions of the aeroelastic behavior of JW and TBW.

The reduction of the structural model would greatly reduce the computational cost: the linear systems that have to be solved during the Newton Raphson iterative method would involve a very small number of generalized coordinates with potentially large CPU time savings. For an Updated Lagrangian Formulation based on a geometrically nonlinear structural triangular shell element with modally reduced linear aerodynamics panel codes it was shown that the results of the reduced order model would greatly
improve with the inclusion of second order modes. For planar cases and non-planar wing systems, it was also demonstrated that updating the modes was more efficient than increasing their number.

A. Contributions of the Present Study

The proposed strategy for modal reduction for both the nonlinear structural model and linear aerodynamic model has the goal of exploiting the findings of a previous work. In particular, it was shown that a small number of modes, if frequently updated, can provide an excellent approximation even for challenging wing configurations with significant structural nonlinearities. Therefore, the idea is to obtain a mathematical description of how the basis of tangent modes changes with the applied aerodynamic pressure. This is accomplished via Proper Orthogonal Decomposition (POD) up to an assigned aerodynamic speed $V_{\text{POD}}$. For each structural tangent mode the POD analysis provides a set of POD coefficients and POD modes (different than the tangent modes) and the variation of each tangent mode with the aerodynamic pressure is reconstructed. The values of the POD coefficients for higher aerodynamic speeds ($V > V_{\text{POD}}$) are calculated by the means of an extrapolation procedure based on the use of Legendre polynomials (function of the load step $\lambda$). The theoretical differences between the present technique and recently proposed available methods are the following:

- A relatively small number of modes is reconstructed and updated by using a set of constant POD modes and load-step-dependent POD coefficients.
- The main computational effort of the proposed method is in the generation of structural tangent modes (several eigenvalue problems need to be solved) and in the POD analysis required to generate the constant POD modes and the variable POD coefficients. However, the quite large number of expensive nonlinear static analyses is not required: the procedure presented in this work does not adopt the quadratic and cubic modal stiffness coefficients. In both the present and existing approaches the creation of the basis of modes require a preparation procedure which is performed once and this is a quite computationally demanding phase.

The paper is organized as follows:

- **Section II: Modal Decomposition**
  A self-contained review of basic definitions and properties of the Proper Orthogonal Decomposition (POD) is provided.
- **Section III: Solution of the Nonlinear Steady State Equations**
  The nonlinear static aeroelastic equations governing the problem are presented and the iterative solution strategy is outlined.
- **Section IV: Full Order Nonlinear Aeroelastic Analysis and Proper Orthogonal Decomposition Procedure**
  The methodology introduced in this study is presented. How the structural tangent modes are reconstructed via POD technique is shown.
- **Section V: Reduced Order Nonlinear Aeroelastic Analysis - The Steady Case**
  The reduced order system of aeroelastic equations in modal coordinates is obtained and the iterative solution technique is presented.
- **Section VI: Extrapolation**
  The POD coefficients are written as a function of the load step $\lambda$. The functions are extrapolated by using a fitting procedure based on the Least Square Method.
- **Section VII: Results**
  The performance of the reduced order aeroelastic model is assessed for both planar and non-planar cases. A delta wing and two challenging Joined Wings are investigated.
- **Section VIII: Conclusions**
  The main significant findings are summarized and discussed. Future possible studies are also presented.
II. Modal Decomposition

In this section we provide a self-contained review of basic definitions and properties of the Proper Orthogonal Decomposition (POD) technique relevant to this work and discuss how the method can be applied to computer simulations in order to separate spatial and temporal behavior. The POD is a well-known technique for determining an optimal basis for the reconstruction of a data set. It has been used in various disciplines that include fluid mechanics, identification and control in chemical engineering, oceanography, analysis of cellular flame patterns, and image processing. Depending on the discipline, the POD is also known as Karhunen-Loève decomposition, principal components analysis, singular systems analysis, or singular value decomposition.

A. Theoretical Aspects

Let us consider a sequence of numerical and/or experimental observations represented by scalar functions \( f(x, t_i), i = 1 \ldots N_s \). Without loss of generality, the time-average of the sequence, defined by

\[
\bar{f}(x) = \langle f(x, t_i) \rangle = \frac{1}{N_s} \sum_{i=1}^{N_s} f(x, t_i),
\]

is assumed to be zero. The Proper Orthogonal Decomposition extracts time-independent orthonormal basis functions, \( \phi_k(x) \), and time-dependent orthonormal amplitude coefficients, \( a_k(t_i) \), such that the reconstruction

\[
f(x, t_i) = \sum_{k=1}^{N_s} a_k(t_i) \phi_k(x), \quad i = 1, \ldots, N_s
\]

is optimal in the sense that the average least squares truncation error

\[
\varepsilon_m = \left\| \sum_{k=1}^{m} a_k(t_i) \phi_k(x) - \sum_{k=1}^{N_s} a_k(t_i) \phi_k(x) \right\|^2
\]

is minimized for any given number \( m \leq N_s \) of basis functions over all possible sets of orthogonal functions. Here \( \| \cdot \| \) is the \( L^2 \)-norm, \( \| f \|^2 = \langle f, f \rangle \), where \( \langle \cdot, \cdot \rangle \) denotes the standard Euclidean inner product; \( \langle \cdot \rangle \) denotes an average operation, usually over time; and the functions \( \phi_k(x) \) are called empirical eigenfunctions, coherent structures, or POD modes. In practice the state of a numerical model is only available at discrete spatial grid points, so that the observations that form the data set are vectors rather than continuous functions. In other words, \( D = (x_1, x_2, \ldots, x_N) \), where \( x_j \) is the \( j \)-th grid point and \( f \) is a vector with components \( f_i = [f(x_1, t_i), f(x_2, t_i), \ldots, f(x_N, t_i)]^T \). The data set can be obtained from numerical simulation, experimental investigation or a combination of the numerical and experimental results. More importantly, it can be shown that the eigenfunctions \( \phi_k \) are the eigenvectors of the tensor product matrix

\[
S(x, y) = \frac{1}{N_s} \sum_{i=1}^{N_s} f_i f_i^T.
\]

B. Computational Implementation: Method of Snapshots

A popular technique for finding the eigenvectors of \( S \) is the method of snapshots developed by Sirovich. It was introduced as an efficient method when the resolution of the spatial domain \( N \) is higher than the number of observations \( (N_s) \). The method of snapshots is based on the fact that the data vectors, \( f_i \), and the eigenvectors \( \phi_k \), span the same linear space. This implies that the eigenvectors can be written as a linear combination of the data vectors

\[
\phi_k = \sum_{i=1}^{N_s} v_i^k f_i, \quad k = 1 \ldots N_s.
\]

After substitution in the eigenvalue problem, \( S(x, y)\phi(y) = \sigma \phi(x) \), the coefficients \( v_i^k \) are obtained from the solution of

\[
C v = \sigma v,
\]
In physical terms, if reconstruction increases to $N$, energy captured by the respective POD mode, $\sigma$, speaking, decomposition of the form is defined as the sum of all eigenvalues of each mode to the overall dynamics. Typically, the modes obtained from the POD technique are optimal spatial features, also known as empirical.

C. Properties of the POD Decomposition

Since the kernel is Hermitian, $S(\mathbf{x}, \mathbf{y}) = S^*(\mathbf{y}, \mathbf{x})$, according to Riesz Theorem\textsuperscript{46} it admits a diagonal decomposition of the form

$$S(\mathbf{x}, \mathbf{y}) = \sum_{k=1}^{N} \sigma_k \phi_k(\mathbf{x}) \phi_k^*(\mathbf{y}).$$

This fact is particularly useful when finding the POD modes analytically. They can be read off from the diagonal decomposition\textsuperscript{7}. Then the temporal coefficients, $a_k(t_i)$, are calculated by projecting the data set on each of the eigenfunctions

$$a_k(t_i) = (f(\mathbf{x}, t_i), \phi_k(\mathbf{x})), \quad i = 1, \ldots, N_s.$$ 

It can be shown that both temporal coefficients and eigenfunctions are uncorrelated in time and space, respectively\textsuperscript{36, 43–45}. In addition, the POD modes $\{\phi_k(\mathbf{x})\}$ and the corresponding temporal coefficients, $\{a_k(t_i)\}$, satisfy the following orthogonality properties

(i) $\phi_j^*(\mathbf{x}) \phi_k(\mathbf{x}) = \delta_{jk}$

(ii) $\langle a_j(t_i) a_k^*(t_i) \rangle = \delta_{jk} \sigma_j$

where $\delta_{jk}$ represents the Kronecker delta function.

Property (ii) is obtained when the terms in the diagonal decomposition are compared with the expression $S(\mathbf{x}, \mathbf{y}) = \sum (a_j(t_i) a_k^*(t_i)) \phi_j(\mathbf{x}) \phi_k^*(\mathbf{y})$. The non-negative and self-adjoint properties of $S(\mathbf{x}, \mathbf{y})$ imply that all eigenvalues are non-negative and can be ordered accordingly: $\sigma_1 \geq \sigma_2 \ldots \geq \sigma_s \geq 0$. Statistically speaking, $\sigma_k$ represents the variance of the data set in the direction of the corresponding POD mode, $\phi_k(\mathbf{x})$. In physical terms, if $f$ represents a component of a velocity field, then $\sigma_k$ measures the amount of kinetic energy captured by the respective POD mode, $\phi_k(\mathbf{x})$. In this sense, the energy measures the contribution of each mode to the overall dynamics.

The total energy captured in a proper orthogonal decomposition of a numerical or experimental data set is defined as the sum of all eigenvalues

$$E = \sum_{k=1}^{N_s} \sigma_k.$$ 

The relative energy captured by the $k^{th}$ mode, $E_k$, is defined by

$$E_k = \frac{\sigma_k}{\sum_{j=1}^{N_s} \sigma_j}.$$ 

Note that the cumulative sum of relative energies, $\sum E_k$, approaches one as the number of modes in the reconstruction increases to $N_s$. The relative phase-angle between two POD-time coefficients is defined by

$$\theta_{ij} = \arctan \left( \frac{a_i}{a_j} \right).$$

D. Interpolation and Extrapolation via POD Decomposition

Typically, the modes obtained from the POD technique are optimal spatial features, also known as empirical eigenfunctions, that can best reconstruct “time”-varying phenomena with additional spatial characteristics. The adjective empirical refers to the fact that no knowledge of the geometry or scales (both in space and
time) are required a priori. Indeed, only the ensemble of data is needed and, thus, the role of space and time can be interpreted rather loosely. That is, the same procedure can be applied over “parameter”-varying data sets by replacing the role of “time” in the standard POD technique with a single parameter of interest. This approach to study spatio-temporal phenomena over parameter space has been employed by Epureanu et al., in the analysis of reduced order models of potential flows in turbomachinery and by Bui-Thanh et al., in their study of transonic aerodynamics. We employ a similar approach in this manuscript to study how the structural tangent modes change with the applied aerodynamic pressure and load step \( \lambda \). Following, we outline next the basic ideas.

Let now \( f(x, t, \lambda_i), i = 1 \ldots N_\lambda \) represent the ensemble of snapshots corresponding to the different set of load steps \( \lambda_1, \ldots, \lambda_{N_\lambda} \). First we perform the POD over the ensemble to extract POD modes \( \phi_k(x) \), and parameter-dependent amplitude coefficients, \( a_k(\lambda_i) \), such that the reconstruction

\[
f(x, \lambda_i) = \sum_{k=1}^{N_\lambda} a_k(\lambda_i) \phi_k(x), \quad i = 1, \ldots, N_\lambda
\]

is still optimal over the entire range of parameter values \( \lambda_1, \ldots, \lambda_{N_\lambda} \). Assuming that the coefficients \( a_k(\lambda_i) \) are smooth functions of the load \( \lambda \), it might be possible possible to interpolate (and extrapolate) at values of \( \lambda \) that are not part of the original ensemble. Extrapolation over a large region of parameter space can be more difficult to realize specially if the ensemble has no periodicity or asymptotic behavior on that particular region. In this work we employ Legendre polynomials to extrapolate through the load steps. The reconstructed set can be written as

\[
f(x, \lambda_i) = \sum_{k=1}^{N_\lambda} L_{lk}(\lambda_i) \phi_k(x), \quad i = 1, \ldots, N_\lambda
\]

where \( L_{lk} \) is the Legendre polynomial of order \( l \) extrapolated from the POD coefficient \( a_k \).

### III. Solution of the Nonlinear Steady State Equations

Before the discussion of the Newton-Raphson solution procedure a brief terminology should be introduced.

“Tangent matrix” defines the matrix which multiplies the unknown displacement vector. Since in the presented case there is the coupling of aerodynamic and structural mathematical models, the “aerodynamic tangent matrix” and the “structural tangent matrix” can be identified. The summation of the two above mentioned matrices is the tangent matrix of the aeroelastic system.

The wing is loaded by external aerodynamic loads, motion dependent aerodynamic loads, and other loads (indicated with \( P_{ext} \)) such as the inertial loads. The Newton Raphson solution procedure used proceeds as follows: The reference aerodynamic pressure \( L_{ref} \) is first calculated. This is the increment in dynamic pressure from one load step to another.

\[
L_{ref} = \frac{1}{2} \rho_\infty V_\infty^2 \frac{N_{step}}{N_{step}}.
\]

An increment of external concentrated loads can similarly be defined. The reference magnitude \( P_{ref} \) of that load will be

\[
P_{ref} = \frac{1}{N_{step}},
\]

where \( P_{ref} \) is a dimension-less number.

Note that because aerodynamic influence coefficients are calculated for a configuration with zero angles of attack for all aerodynamic boxes and because this is the geometry from which motion starts, no motion dependent aerodynamic loads appear initially, and unless there is a non aerodynamic external load the structure will not move. At the very first iteration of the Newton Raphson procedure an initial angle of attack perturbation is imposed:

\[
x_{\text{step 1 iter 1}} = x_{\text{pert}} \neq x_{\alpha=0}
\]

Considering this perturbation of the system, the cumulative displacement vector is initialized:

\[
x_{\text{step 1 iter 1}} - x_{\alpha=0} = U^0
\]
Mathematically equation (17) is not correct and the RHS and LHS of the equation are not equivalent. However, as explained in a previous work21 when these quantities are multiplied by the aerodynamic matrix then the equivalence is correct. So, the initialization (17) can be done without introducing errors and with a great simplification of the theory.

The applied non-aerodynamic loads (of the non-follower force type) are only step dependent and they are calculated by using the following expression:

\[
P_{\text{str}}^{\text{step} \lambda} = \lambda \cdot P_{\text{ref}} \cdot P_{\text{ext}}
\]  

The aerodynamic loads are calculated (at the very first iteration \( U_{\text{step} \lambda \text{iter}(n-1)} = U^{(0)} \)):

\[
L_{\text{RHS}}^{\text{step} \lambda \text{iter} n} = \lambda L_{\text{ref}} C U_{\text{step} \lambda \text{iter}(n-1)},
\]  

where \( \lambda \) is the load factor and it is equal to 1 for the first load step, 2 for the second load step, and so forth. The internal forces \( F_{\text{int}}^{\text{step} \lambda \text{iter} n} \) are known from the previous iteration (if the very first iteration of the first load step is considered, there are no internal forces because the structure is initially assumed to be stress-free). So the unbalanced loads \( P_{\text{unb}}^{\text{step} \lambda \text{iter} n} \) can be calculated:

\[
P_{\text{unb}}^{\text{step} \lambda \text{iter} n} = P_{\text{str}}^{\text{step} \lambda} + P_{\text{RHS}}^{\text{step} \lambda \text{iter} n} - F_{\text{int}}^{\text{step} \lambda \text{iter} n}
\]  

The structural tangent matrix \( K_{T}^{\text{step} \lambda \text{iter} n} \) is calculated by adding the elastic stiffness matrix \( K_{E}^{\text{step} \lambda \text{iter} n} \) (calculated considering the coordinates at the beginning of the \( n \text{th} \) iteration) and the geometric stiffness matrix \( K_{G}^{\text{step} \lambda \text{iter} n} \). In practice it is convenient to perform this operation at element level and then assemble the resulting matrix.

\[
K_{T}^{\text{step} \lambda \text{iter} n} = K_{E}^{\text{step} \lambda \text{iter} n} + K_{G}^{\text{step} \lambda \text{iter} n}
\]

The structural tangent matrix is updated at each iteration of the procedure. The term iteration used here refers to the repetitive refinement of a nonlinear solution for an incremental load step. It does not refer to the process of increasing loads and dynamic pressure incrementally.

The aerodynamic tangent matrix is calculated by:

\[
K_{\text{Taero}}^{\text{step} \lambda} = -\lambda L_{\text{ref}} C.
\]  

Note that the aerodynamic tangent matrix is only load step dependent. The reason is that the matrix \( C \) is constant. The matrix is constant because of the linearity of the aerodynamic theory and the assumption that linearized aerodynamic loads calculated on a reference grid at the beginning of a load step will provide accurate aerodynamic loads for the process of converging, at a load increment, on the incremental deformation solution. Because results are presented here for which dynamic pressure is graduatedly increased, the procedure is valid when the flow can be considered incompressible, or when dynamic pressure is increased at any other non-zero constant Mach number. If the Mach number is changed and the hypothesis of incompressible flow is removed, then the matrix \( C \) is not constant and it becomes load step dependent. In fact, when the convergence of a particular load step is reached, the load step is incremented by one and the dynamic pressure (and so the speed) is incremented as well. This increase means a different Mach number and so a different matrix \( C \) is calculated by the DL code. In this paper this is not the case and all the results will assume incompressible flow and constant aerodynamic matrix \( C \) calculated once.

Now, the tangent matrix \( K_{\text{Tang}}^{\text{step} \lambda \text{iter} n} \) is built by adding the structural and aerodynamic tangent matrices as follows:

\[
K_{\text{Tang}}^{\text{step} \lambda \text{iter} n} = K_{T}^{\text{step} \lambda \text{iter} n} + K_{\text{Taero}}^{\text{step} \lambda}.
\]

The following linear system can now be solved, and the displacement vector \( u_{\text{Tang}}^{\text{step} \lambda \text{iter} n} \) can be found:

\[
K_{\text{Tang}}^{\text{step} \lambda \text{iter} n} \cdot u_{\text{Tang}}^{\text{step} \lambda \text{iter} n} = P_{\text{unb}}^{\text{step} \lambda \text{iter} n}.
\]

Node location coordinates are updated for the next iteration:

\[
x_{\text{iter} (n+1)} = x_{\text{iter} n} + u_{d}^{\text{step} \lambda \text{iter} n},
\]
where $u_{n}^{\text{step} \lambda \text{iter} n}$ is the vector which contains only the translational degrees of freedom, and it is obtained from the vector of displacements $u_{n}^{\text{step} \lambda \text{iter} n}$ by eliminating the rows corresponding to the rotations. If the last iteration of the load step $\lambda$ has been performed, then the left hand side of the previous equation is $x_{n}^{\text{step} (\lambda+1) \text{iter} 1}$ instead of $x_{n}^{\text{step} \lambda \text{iter} (n+1)}$.

Rigid body motion is eliminated from elements according to the Levy-Gal’s procedure and the pure elastic rotations and strains are found. Using these quantities the internal forces are updated for the next iteration and, therefore, the vector $F_{\text{int}}^{\text{step} \lambda \text{iter} (n+1)}$ is created (in the case in which the last iteration of load step $\lambda$ has been performed the term $F_{\text{int}}^{\text{step} \lambda \text{iter} (n+1)}$ has to be replaced by $F_{\text{int}}^{\text{step} (\lambda+1) \text{iter} 1}$).

The cumulative displacement vector is updated next:

$$U_{\text{step} \lambda \text{iter} n} = U_{\text{step} \lambda \text{iter} (n-1)} + u_{\text{step} \lambda \text{iter} n}.$$  

(26)

The procedure is repeated until a desired convergence tolerance is reached. More details on this formulation can be found in Ref. [21]

### IV. Full Order Nonlinear Aeroelastic Analysis and Proper Orthogonal Decomposition Procedure

The number of low-frequency structural tangent modes used in the Reduced Order Model (ROM) is indicated with $\mathcal{N}$. The full order nonlinear analysis is then started. At the beginning of each load step the first $\mathcal{N}$ tangent modes are calculated. These modes are in general different than the modes calculated at previous load steps because of the presence of structural nonlinearity. Let $\Phi_i^{\lambda}(\lambda)$ be the $i$th structural tangent mode. $\lambda$ indicates the generic load step. At each load step all the tangent modes are normalized to make the mass matrix an identity matrix:

$$\Phi_i^{\lambda}(\lambda)^T \cdot M \cdot \Phi_i^{\lambda}(\lambda) = 1$$  

(27)

Each node has 6 degrees of freedom associated with it. The quantities $u_1$, $u_2$, and $u_3$ indicate the finite element translational displacements in the $x$, $y$ and $z$ directions respectively. The variables $u_4$, $u_5$, $u_6$ represent the finite element rotations. To explain the notation consider a simple case represented by a structural model with just three structural nodes $N_1$, $N_2$, and $N_3$. For that particular case, the $i$th normalized tangent mode is given by the following vector:

$$\Phi_i^{\lambda} = \begin{bmatrix} u_1^{N_1} & u_2^{N_1} & u_3^{N_1} & u_4^{N_1} & u_5^{N_1} & u_6^{N_1} \\ u_1^{N_2} & u_2^{N_2} & u_3^{N_2} & u_4^{N_2} & u_5^{N_2} & u_6^{N_2} \\ u_1^{N_3} & u_2^{N_3} & u_3^{N_3} & u_4^{N_3} & u_5^{N_3} & u_6^{N_3} \end{bmatrix}$$  

(28)

A vector obtained by extracting only one component (for example $u_1$) of all the nodes can be obtained. Using the example of a structure with 3 nodes it is possible to write from equation (28):

$$u_1 \Phi_i^{\lambda} = \begin{bmatrix} u_1^{N_1} \\ u_1^{N_2} \\ u_1^{N_3} \end{bmatrix}$$  

(29)

Vectors of the type $u_2 \Phi_i^{\lambda}$, $u_3 \Phi_i^{\lambda}$ etc. can be defined in a similar fashion. For example, for the case of $u_3 \Phi_i^{\lambda}$ and 3 structural nodes the following relation is valid (see equation (28)):

$$u_3 \Phi_i^{\lambda} = \begin{bmatrix} u_3^{N_1} \\ u_3^{N_2} \\ u_3^{N_3} \end{bmatrix}$$  

(30)

As explained in the example of a structure with 3 nodes, in the general case the vectors of the type $u_i \Phi_i^{\lambda}(\lambda)$ have $N_N$ entries ($N_N$ is the number of FE nodes) and are extracted from $\Phi_i^{\lambda}(\lambda)$. At each load step $\lambda$ the modes are stored for the Proper Orthogonal Decomposition (POD) analysis. The POD is performed for each of the vectors of the type $u_i \Phi_i^{\lambda}(\lambda)$ where $i = 1, 2, ..., \mathcal{N}$ and $j = 1, ..., 6$ (note that each structural node has 6 degrees of freedom). The vectors $u_i \Phi_i^{\lambda}$ are named structural tangent submodes. $\lambda$, the load step, is the free parameter used in the POD analysis. Using the concept expressed by equation [28] the generic structural tangent submode can be reconstructed as

$$u_i \Phi_i^{\lambda}(\lambda) \approx u_i \phi_1^{\lambda} + u_i \phi_2^{\lambda} + u_i \phi_3^{\lambda} + \cdots + u_i \phi_M^{\lambda} + \phi:M \cdot a_M^{\lambda} \lambda = 1, ..., N_{\text{step}} \quad j = 1, ..., 6$$  

(31)

$\mathcal{M}$ is the number of POD modes used to reconstruct $u_i \Phi_i^{\lambda}(\lambda)$. $\mathcal{M}$ can not be larger than the number $N_{\text{step}}$ of load steps: $\mathcal{M} \leq N_{\text{step}}$. The number of POD modes used to reconstruct $u_i \Phi_i^{\lambda}(\lambda)$ can be, in principle, different than the number used to reconstruct $u_k \Phi_k^{\lambda}(\lambda)$ when $k \neq j$. In this work $\mathcal{M} = \text{const} \leq N_{\text{step}}$.

$u_i \phi_1^{\lambda}$, $u_i \phi_2^{\lambda}$, ... are the POD modes. $u_i a_1^{\lambda}(\lambda)$, $u_i a_2^{\lambda}(\lambda)$, ... are the POD coefficients which are function of the parameter $\lambda$. The POD modes are constant and do not change with $\lambda$. 

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V. Reduced Order Nonlinear Aeroelastic Analysis - The Steady Case

At the first iteration of each load step \( \lambda \) the structural tangent submodes \( u^1 \Phi^i (\lambda), u^2 \Phi^i (\lambda), u^3 \Phi^i (\lambda), u^4 \Phi^i (\lambda), u^5 \Phi^i (\lambda), \) and \( u^6 \Phi^i (\lambda) \) are updated by using equation (31). For example, the submode \( u^3 \Phi^i (\lambda) \) is updated as follows:

\[
u^3 \Phi^i (\lambda) \approx u^3 \phi^i_1 + u^3 \phi^i_2 + u^3 \phi^i_3 + ... + u^3 \phi^i_M \cdot u_j a^i_j (\lambda) \quad \lambda = 1, ..., N_{\text{step}}
\]  

(32)

The \( i^{th} \) structural tangent mode \( \Phi^i \) is built by assembling the structural tangent submodes \( u^1 \Phi^i (\lambda), u^2 \Phi^i (\lambda), u^3 \Phi^i (\lambda), u^4 \Phi^i (\lambda), u^5 \Phi^i (\lambda), \) and \( u^6 \Phi^i (\lambda) \) previously reconstructed. The transformation matrix \( \Psi \), which contains the first \( N \) low-frequency tangent modes is created as follows:

\[
\Psi (\lambda) = \begin{bmatrix}
\Phi^1 (\lambda) & \Phi^2 (\lambda) & \Phi^3 (\lambda) & \Phi^i (\lambda) & \Phi^{i+1} (\lambda) & \ldots & \Phi^{N} (\lambda)
\end{bmatrix}
\]  

(33)

The transformation matrix defined in equation (33) is kept constant (i.e. is not updated) for all the iterations within load step \( \lambda \). It is updated only when a new load step is considered (see equation (32)). From now on the notation \( \Psi^{\text{step} \lambda \text{iter} n} \equiv \Psi (\lambda) \) will be used for consistency with the FEM notation. At each iteration the full order aeroelastic tangent matrix \( K^{\text{step} \lambda \text{iter} n} \) is built. The full order problem that has to be solved is the following:

\[
K^{\text{step} \lambda \text{iter} n} \cdot u^{\text{step} \lambda \text{iter} n} = P^{\text{step} \lambda \text{iter} n}_{\text{unb}}
\]  

(34)

The following modal reduction is performed:

\[
u^{\text{step} \lambda \text{iter} n} \approx \Psi^{\text{step} \lambda} q^{\text{step} \lambda \text{iter} n}
\]  

(35)

and pre-multiplying the equation by the transpose of matrix \( \Psi^{\text{step} \lambda} \) it is possible to obtain:

\[
\Psi^{\text{step} \lambda} T \cdot K^{\text{step} \lambda \text{iter} n} \cdot \Psi^{\text{step} \lambda} q^{\text{step} \lambda \text{iter} n} = \Psi^{\text{step} \lambda} T \cdot P^{\text{step} \lambda \text{iter} n}_{\text{unb}}
\]  

(36)

or

\[
K^{\text{step} \lambda \text{iter} n}_{\text{Tangent\ Red}} \cdot q^{\text{step} \lambda \text{iter} n} = P^{\text{step} \lambda \text{iter} n}_{\text{unb\ Red}}
\]  

(37)

where \( K^{\text{step} \lambda \text{iter} n}_{\text{Tangent\ Red}} \) represents the aeroelastic reduced order tangent matrix; \( P^{\text{step} \lambda \text{iter} n}_{\text{unb\ Red}} \) is the reduced order vector of unbalanced forces and \( q^{\text{step} \lambda \text{iter} n} \) are the unknown modal generalized coordinates. The solution of equation (37) leads to the finding of vector \( q^{\text{step} \lambda \text{iter} n} \). Finally, the full order FE displacement vector \( u^{\text{step} \lambda \text{iter} n} \) can be found by using relation (35). With the knowledge of the displacement vector all the other quantities, such as cumulative displacement vector, vector of the coordinates of the nodes and vector of the internal forces can be calculated for the next iteration. The rigid body rotation is removed after the full order FE displacement vector is determined. At that point the coordinates are updated and a new iteration is performed. If a new iteration corresponds to a new load step then the basis of structural tangent modes is updated by using the procedure already described (see for example equation (32) for the submode \( u^3 \Phi^i \) relative to the \( i^{th} \) structural tangent mode).

VI. Extrapolation

The POD reconstruction is valid up to load step \( \lambda = N_{\text{step}} \) which corresponds to a speed \( V = V_{\text{POD}} \). The question is now the following: is it possible to extrapolate and perform the reduction by using POD modes to reconstruct the tangent modes and still have good accuracy? Extrapolation means that the parameter \( \lambda \) can be larger that \( N_{\text{step}} \) and so \( V > V_{\text{POD}} \). To answer this question, the generic contribution \( u^j \Phi^i \) to the tangent model \( \Phi^i \) is considered:

\[
u^j \Phi^i (\lambda) \approx u^j \phi^i_1 + u^j \phi^i_2 + u^j \phi^i_3 + ... + u^j \phi^i_M \cdot u_j a^i_j (\lambda) \quad \lambda = 1, ..., N_{\text{step}}
\]  

(38)

Now focus on \( u^j a^i_j (\lambda) \). This POD coefficient changes with the load step \( \lambda \). A vector whose entries are the values of the coefficient for the different values of \( \lambda \) can be defined as follows:

\[
u_j a^i_j = \begin{bmatrix}
u^j a^i_j (\lambda = 1) \nu^j a^i_j (\lambda = 2) \ldots \nu^j a^i_j (\lambda = N_{\text{step}})
\end{bmatrix}^T
\]  

(39)
Figure 1. POD coefficient $u_1 a_{i}^{s}(\lambda)$ and its modulus vs load step $\lambda$. No extrapolation is performed. Case of delta wing; $V_{\infty} = 21$ m/s.

Figure 2. POD coefficient $u_1 a_{i}^{s}(\lambda)$ and its modulus vs load step $\lambda$. No extrapolation is performed. Case of delta wing; $V_{\infty} = 21$ m/s.

Similar definitions can be adopted for the other POD coefficients. As previously outlined, it is possible to demonstrate that any two distinct vectors of the type described in equation 39 are orthogonal:

$$u_j a_{i}^{s} T \cdot u_j a_{i}^{t} = 0 \quad \text{if } s \neq t$$

(40)

This orthogonality condition will not be imposed in the extrapolation process.

The coefficients of the type $u_j a_{i}^{s}(\lambda)$ (see equation 38) can be seen as a function of the load step $\lambda$. These functions are quite difficult to approximate as demonstrated in Figures 1a), 2a), 3a), and 4a) which report some of the coefficients for the case of the delta wing discussed below. If the absolute values of the coefficients are considered, their functional relationship with the parameter $\lambda$ is much simpler (see Figures 1b), 2b), 3b), and 4b)). This empirical observation suggests that instead of interpolating the actual coefficients $u_j a_{i}^{s}(\lambda)$ it is more convenient and practical to interpolate their corresponding absolute values $|u_j a_{i}^{s}(\lambda)|$.

The functions representing the variation of the absolute values of the POD coefficients with the load step $\lambda$ are assumed to be Legendre polynomials. Legendre polynomials are defined in the interval $[-1,1]$. Therefore, the variables need to be changed to satisfy this requirement. Suppose that the desired value of the maximum extrapolated $\lambda$ is $\lambda_{\text{max}}$ (note that this is an integer number): $\lambda_{\text{max}} > N_{\text{step}}$. The following transformation of variables is assumed:

$$\zeta = c_1 \lambda + c_2 \quad -1 \leq \zeta \leq +1; \quad 1 \leq \lambda \leq \lambda_{\text{max}}$$

(41)

When $\lambda = \lambda_1 = \lambda_{\text{min}} = 1$ then $\zeta$ has to be $-1$ and when $\lambda = \lambda_{\text{max}}$ then $\zeta$ has to be $+1$. This implies that

$$-1 = c_1 + c_2$$

$$+1 = c_1 \lambda_{\text{max}} + c_2$$

(42)
From which it follows

\[ c_1 = \frac{2}{\lambda_{\text{max}} - 1} \]
\[ c_2 = -\frac{\lambda_{\text{max}} + 1}{\lambda_{\text{max}} - 1} \]

The transformation is then the following:

\[ \zeta = \frac{2\lambda}{\lambda_{\text{max}} - 1} - \frac{\lambda_{\text{max}} + 1}{\lambda_{\text{max}} - 1} \quad -1 \leq \zeta \leq +1; \quad 1 \leq \lambda \leq \lambda_{\text{max}} \]  

The number \( \mathcal{L} \) of Legendre polynomials used in the approximation is decided a priori. The following fitting, for the absolute values of all the entries of vector \( u_j a_i^s \), is used:

\[
\begin{cases}
|u_j a_i^s(\zeta_1)| &= u_j A_{s_0} L_0(\zeta_1) + u_j A_{s_1} L_1(\zeta_1) + \ldots + u_j A_{s_L} L_L(\zeta_1) \quad \lambda = 1 \iff \zeta = \zeta_1 \\
|u_j a_i^s(\zeta_2)| &= u_j A_{s_0} L_0(\zeta_2) + u_j A_{s_1} L_1(\zeta_2) + \ldots + u_j A_{s_L} L_L(\zeta_2) \quad \lambda = 2 \iff \zeta = \zeta_2 \\
&\ldots \\
|u_j a_i^s(\zeta_{N_{\text{step}}})| &= u_j A_{s_0} L_0(\zeta_{N_{\text{step}}}) + u_j A_{s_1} L_1(\zeta_{N_{\text{step}}}) + \ldots + u_j A_{s_L} L_L(\zeta_{N_{\text{step}}}) \quad \lambda = N_{\text{step}} \iff \zeta = \zeta_{N_{\text{step}}}
\end{cases}
\]

where for example \( \zeta_{N_{\text{step}}} \) is the variable \( \zeta \) corresponding to the maximum value of \( \lambda \) used to perform the POD analysis:

\[ \zeta_{N_{\text{step}}} = \frac{2N_{\text{step}}}{\lambda_{\text{max}} - 1} - \frac{\lambda_{\text{max}} + 1}{\lambda_{\text{max}} - 1} \]  

\( L_0, L_1, \ldots, L_L \) are Legendre polynomials. The unknowns of equation (45) are the coefficients of the Legendre polynomials: \( u_j A_{s_0}^i, u_j A_{s_1}^i \) and so on. The number of unknowns is equal to the number \( \mathcal{L} \) of Legendre
polynomials used in the approximation. The number of equations is equal to the number $N_{\text{step}}$ of steps used in the nonlinear analysis that generated the POD modes. Note that in general the number of equations $N_{\text{step}}$ is quite larger if compared to the number $\mathcal{L}$ of unknowns. The system of equations \ref{eq:7a} can be written in a compact form:

$$u^j a_{s}^i = L \cdot u^j A_{s}^i$$

(47)

where $u^j a_{s}^i$ is a vector which contains the absolute values of the coefficients relative to the $s$th POD mode used to reconstruct the structural tangent submode $u^j \Phi^i$. That vector has $N_{\text{step}}$ entries. $L$ is a $N_{\text{step}} \times \mathcal{L}$ rectangular matrix which contains the Legendre polynomials evaluated at each load step. The $u^j A_{s}^i$ vector contains the unknown coefficients that have to be determined via Least Square Method. Equation \ref{eq:7a} can be elaborated as follows:

$$u^j a_{s}^i = L \cdot u^j A_{s}^i \Rightarrow L^T \cdot u^j a_{s}^i = L^T \cdot L \cdot u^j A_{s}^i \Rightarrow \Rightarrow u^j A_{s}^i = \left[ L^T \cdot L \right]^{-1} \cdot L^T \cdot u^j a_{s}^i$$

(48)

where $L^T$ indicates the transpose of matrix $L$. At this stage the coefficients stored in the $u^j A_{s}^i$ vector are known and it is possible to calculate the new extrapolated values. In detail, it is possible to write:

$$\begin{align*}
|u^j a_{s}^i(\zeta_{N_{\text{step}}+1})| = & u^j A_{s}^i L_0 (\zeta_{N_{\text{step}}+1}) + u^j A_{s}^i L_1 (\zeta_{N_{\text{step}}+1}) + \cdots + u^j A_{s}^i L_{\mathcal{L}} (\zeta_{N_{\text{step}}+1}) \\
|u^j a_{s}^i(\zeta_{N_{\text{step}}+2})| = & u^j A_{s}^i L_0 (\zeta_{N_{\text{step}}+2}) + u^j A_{s}^i L_1 (\zeta_{N_{\text{step}}+2}) + \cdots + u^j A_{s}^i L_{\mathcal{L}} (\zeta_{N_{\text{step}}+2}) \\
|u^j a_{s}^i(\zeta_{N_{\text{step}}+3})| = & u^j A_{s}^i L_0 (\zeta_{N_{\text{step}}+3}) + u^j A_{s}^i L_1 (\zeta_{N_{\text{step}}+3}) + \cdots + u^j A_{s}^i L_{\mathcal{L}} (\zeta_{N_{\text{step}}+3}) \\
\vdots & \\
|u^j a_{s}^i(\zeta_{\lambda_{\text{max}}})| = & u^j A_{s}^i L_0 (\zeta_{\lambda_{\text{max}}}) + u^j A_{s}^i L_1 (\zeta_{\lambda_{\text{max}}}) + \cdots + u^j A_{s}^i L_{\mathcal{L}} (\zeta_{\lambda_{\text{max}}})
\end{align*}$$

(49)

The Least Square procedure here described for the vector $u^j a_{s}^i$ is used for all the vectors containing the POD coefficients.

After the extrapolated absolute values for the coefficients are found, the actual coefficients are obtained with the following algorithm:

- If the extrapolated absolute value of the corresponding coefficient is positive, then the coefficient is assumed to have that value. Note that this approximation will not enforce the orthogonality of the coefficients (see equation \ref{eq:7a}) which is, in general, lost.
- If the extrapolated absolute value of the corresponding coefficient is negative, then this is clearly not an acceptable result. Therefore, the corresponding coefficient is set to be zero.

The previous extrapolation procedure can be explained with an example. Suppose that the coefficient corresponding to tangent mode # 17, component $u_3$ and POD mode # 4 is under consideration. If this is the case, the coefficient that is considered is $u^3 a_{17}^4$. A fitting process is performed and the values of $u^3 a_{17}^4$ for load steps higher than $N_{\text{step}}$ (i.e., the extrapolation is performed) are obtained. If $|u^3 a_{17}^4(\zeta)| \leq 0$ then it is assumed $u^3 a_{17}^4(\zeta) = 0$. If $|u^3 a_{17}^4(\zeta)| \geq 0$ then it is assumed $u^3 a_{17}^4(\zeta) = |u^3 a_{17}^4(\zeta)|$.

A second extrapolation option is now described. For speeds higher than $V_{\text{POD}}$ the extrapolation used to reconstruct the structural tangent modes can be performed by using the POD coefficients evaluated at the last non-extrapolated value for the load step. That is to: the coefficients used in the reconstruction of the modes are the ones valid for $\lambda = N_{\text{step}}$. The two presented methods will be compared for planar (delta wing) and non-planar cases (joined wings).

VII. Results

A. Validation of the Present Aeroelastic Capability

Figures \ref{fig:1} and \ref{fig:2} present the static linear and nonlinear aeroelastic analyses of the Joined Wing reported in Figure \ref{fig:3}. It is possible to see the excellent correlation with NASTRAN (linear static aeroelastic analysis) and the strong effects of the structural nonlinearity: for an assigned aerodynamic speed of 50 [m/s], a
selected point on the lower wing (Figure 5a)) presents a larger displacement in the nonlinear case; the reverse happens for a point on the upper wing (Figure 5b)). This example shows the important role of the structural nonlinearity for this type of wing configurations. The present full order nonlinear aeroelastic capability has also been validated for static and dynamic cases in previous studies.20, 21, 33, 50–52

Figure 5. Linear and nonlinear static aeroelastic deflection of a Joined Wing (see Figure 6): validation and effects of the geometric nonlinearity. $P_1 \equiv (a, 6a, 0)$; $P_4 \equiv (29a, 14a, 43a)$; $a = 50\text{ mm}$.

Figure 6. Joined wing model II. Joint located at 70% of the wing span. $h_1 = h_2 = h_3 = 2\text{ mm}; h_4 = 0.5\text{ mm}$

B. Proof of Concept

The wing systems analyzed in this work are a delta wing and 2 joined wings represented in Figures 6 and 7. It is demonstrated that for both planar and challenging non-planar configurations the use of structural tangent modes updated at each load step provides excellent approximation of the full order solution. Figure 8 shows the comparison between the full order solution and the reduced order solution formulated with the use of structural tangent modes previously generated and stored during the full order analysis. This suggests
that a basis built by taking the first low frequency modes is sufficient to capture well the main nonlinear effects. The key of the accuracy is in the updating of the basis. The POD analysis gives an analytical expression (see equation (31)) which may be extrapolated for different values of the dynamic pressures. From Figures 9 and 10 it is confirmed that the same concept is also valid for challenging non-planar cases with important geometric nonlinearities. The studies presented in Figures 8-10 do not involve the POD analysis: the structural tangent modes required to build the transformation matrix are calculated and stored when the full order analysis is considered and the reduced order analysis is performed by reading the modes previously stored.

C. Reduced Order Analysis - Planar Case (Delta Wing)

The reduced order model is obtained with a set of 20 structural tangent modes which are reconstructed as a function of the load step $\lambda$ with a set of constant POD modes and coefficients which are functions of the load step. The results presented in Figure 11 show that it is possible to increase the aerodynamic speed for about 10% (which corresponds to an increment of dynamic pressure equal to 21%) and have a very good approximation with a fourth Legendre polynomial ($\mathcal{L} = 5$) extrapolation. Figure 12 presents a convergence test: both the number of POD modes used to reconstruct the structural tangent modes and the number of tangent modes are changed. It is shown that a very few number of POD modes is sufficient to reconstruct the modes correctly. It is also evident that a small number of structural tangent modes is sufficient to have excellent ROM results and a very good extrapolation beyond $V = V_{\text{POD}} = 21m/s$.

Figure 13 presents a comparison with the 2 proposed extrapolation strategies. It is demonstrated that for the case of delta wing the best extrapolation methodology involves the fitting process of the absolute values of the POD coefficients via the adoption of Legendre polynomials.

D. Reduced Order Analysis - Non-planar Case (Joined Wings)

Two types of joined wings are considered. The first one presents the joint at 100% of the wing span (see Figure 7). The second Joined Wing model is derived from the first one and has the joint location at 70%
Figure 8. Static nonlinear aeroelastic analysis. Comparison between the Reduced Order Model (no POD analysis). For the ROM case the basis is continuously updated at each load step by reading the structural tangent modes generated with a previously considered full order analysis. $V_\infty = 21 \frac{m}{s}$, $\alpha = 1/180 \pi$.

Figure 9. Joined wing model I. Static nonlinear aeroelastic analysis. Comparison between the Reduced Order Model (no POD analysis). For the ROM case the basis is continuously updated at each load step by reading the structural tangent modes generated with a previously considered full order analysis. $V_\infty = 30 \frac{m}{s}$, 952 aerodynamic panels). Lower wing, joint and upper wing have thickness $h_L = 2.0$ mm, $h_J = 2.0$ mm and $h_U = 0.5$ mm respectively. $P_1 \equiv (a, 5a, 0)$; $P_2 \equiv (a, 10a, 0)$; $P_3 \equiv (2a, 5a, \frac{3}{2}a)$; $a = 50$ mm.
Figure 10. Joined wing model II. Static nonlinear aeroelastic analysis. Comparison between the Reduced Order Model (no POD analysis). For the ROM case the basis is continuously updated at each load step by reading the structural tangent modes generated with a previously considered full order analysis. \( V_\infty = 30 \text{ m/s} \).

On POD coefficients assumed equal to the coefficients corresponding to load step \( \lambda = N_{\text{step}} \). Moreover, even a quite large number (50) of structural tangent modes is not sufficient to capture the strong nonlinearity beyond \( V = V_{\text{POD}} \): the quality of the reduced order model approximation is lost after an extrapolated value of the aerodynamic speed equal to 5%. These facts are confirmed when a point on the upper wing is considered (see Figures 16 and 17).

As far as the performance of the present reduced order model technique is concerned, the second Joined Wing Model (see Figure 6) does not present substantial differences with respect to the Joined Wing with joint located at 100% of the wing span. This is confirmed for a point on the lower wing (see Figures 18 and 19) and for a point on the upper wing (see Figures 20 and 21).
Figure 11. Case of delta wing. ROM obtained by using 20 tangent modes. The POD analysis is adopted to reconstruct the variation of the structural tangent modes with the load step \( \lambda \). The extrapolation is performed by approximation of the absolute values of the POD coefficients of the type \( u_{ij}(\lambda) \). Fourth order Legendre polynomial and the Least Square Method are used for each coefficient. The full order reference solution is obtained with 50 load steps. The number of POD modes coincides with the number of load steps used to build the basis (30). Note: some of the extrapolated velocity are larger than the consistent flutter speed \( V_F = 24.5 \text{ m/s} \) (dynamic instability); this implies that the extrapolation beyond the flutter speed has only a mathematical meaning in the assessment of the methodology.

Figure 12. Case of delta wing. Convergence study: effect of the number of tangent modes and POD modes used to reconstruct their variation with the load step \( \lambda \). The extrapolation is performed by approximation of the absolute values of the POD coefficients of the type \( u_{ij}(\lambda) \). Fourth order Legendre polynomial and the Least Square Method are used for each coefficient. The full order reference solution is obtained with 50 load steps. Note: some of the extrapolated velocity are larger than the consistent flutter speed \( V_F = 24.5 \text{ m/s} \) (dynamic instability); this implies that the extrapolation beyond the flutter speed has only a mathematical meaning in the assessment of the methodology.
Figure 13. Case of delta wing. Convergence study: effect of the type of extrapolation. For the case in which the extrapolation is based on a fitting process a fourth order Legendre polynomial and the Least Square Method are used for each POD coefficient of the type $u_j^a(l)$. Note: some of the extrapolated velocity are larger than the consistent flutter speed $V_F = 24.5 \text{ m/s}$ (dynamic instability); this implies that the extrapolation beyond the flutter speed has only a mathematical meaning in the assessment of the methodology.

Figure 14. Case of Joined Wing I. Convergence study: effect of the number of tangent modes and POD modes used to reconstruct their variation with the load step $\lambda$. The extrapolation is performed by approximation of the absolute values of the POD coefficients of the type $u_j^a(l)$. Fourth order Legendre polynomial and the Least Square Method are used for each coefficient. The full order reference solution is obtained with 60 load steps. Lower wing, joint and upper wing have thickness $h_L = 2.0 \text{ mm}$, $h_J = 2.0 \text{ mm}$ and $h_U = 0.5 \text{ mm}$ respectively. $P_1 \equiv (a, 5a, 0); a = 50 \text{ mm}$. 

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Case of Joined Wing I. Convergence study: effect of the type of extrapolation. For the case in which the extrapolation is based on a fitting process a fourth order Legendre polynomial and the Least Square Method are used for each POD coefficient of the type \( u_j a_i(\lambda) \). The full order reference solution is obtained with 60 load steps. Lower wing, joint and upper wing have thicknesses \( h_L = 2.0 \, \text{mm} \), \( h_J = 2.0 \, \text{mm} \) and \( h_U = 0.5 \, \text{mm} \) respectively. \( P_1 \equiv (a, 5a, 0); \ a = 50 \, \text{mm} \).

Case of Joined Wing I. Convergence study: effect of the number of tangent modes and POD modes used to reconstruct their variation with the load step \( \lambda \). The extrapolation is performed by approximation of the absolute values of the POD coefficients of the type \( u_j a_i(\lambda) \). Fourth order Legendre polynomial and the Least Square Method are used for each coefficient. The full order reference solution is obtained with 60 load steps. Lower wing, joint and upper wing have thicknesses \( h_L = 2.0 \, \text{mm} \), \( h_J = 2.0 \, \text{mm} \) and \( h_U = 0.5 \, \text{mm} \) respectively. \( P_1 \equiv (2a, 5a, \frac{3}{2}a); \ a = 50 \, \text{mm} \).
Figure 17. Case of Joined Wing I. Convergence study: effect of the type of extrapolation. For the case in which the extrapolation is based on a fitting process a fourth order Legendre polynomial and the Least Square Method are used for each POD coefficient of the type $u_j a_i^l (\lambda)$. The full order reference solution is obtained with 60 load steps. Lower wing, joint and upper wing have thickness $h_L = 2.0 \text{ mm}$, $h_J = 2.0 \text{ mm}$ and $h_U = 0.5 \text{ mm}$ respectively. $P_3 \equiv (2a, 5a, \frac{3}{5}a); a = 50 \text{ mm}$.

Figure 18. Case of Joined Wing II. Convergence study: effect of the number of tangent modes and POD modes used to reconstruct their variation with the load step $\lambda$. The extrapolation is performed by approximation of the absolute values of the POD coefficients of the type $u_j a_i^l (\lambda)$. Fourth order Legendre polynomial and the Least Square Method are used for each coefficient. The full order reference solution is obtained with 60 load steps. $P_1 \equiv (a, 6a, 0); a = 50 \text{ mm}$. 
Figure 19. Case of Joined Wing II. Convergence study: effect of the type of extrapolation. For the case in which the extrapolation is based on a fitting process a fourth order Legendre polynomial and the Least Square Method are used for each POD coefficient of the type $u_{ij}^l(\lambda)$. The full order reference solution is obtained with 60 load steps. \( P_1 = (a, 60, 0); \ a = 50 \text{ mm} \).

Figure 20. Case of Joined Wing II. Convergence study: effect of the number of tangent modes and POD modes used to reconstruct their variation with the load step $\lambda$. The extrapolation is performed by approximation of the absolute values of the POD coefficients of the type $u_{ij}^l(\lambda)$. Fourth order Legendre polynomial and the Least Square Method are used for each coefficient. The full order reference solution is obtained with 60 load steps. \( P_4 = (29, 14, 13, 4, 2); \ a = 50 \text{ mm} \).
Figure 21. Case of Joined Wing II. Convergence study: effect of the type of extrapolation. For the case in which the extrapolation is based on a fitting process a fourth order Legendre polynomial and the Least Square Method are used for each POD coefficient of the type $a_i^j(\lambda)$. The full order reference solution is obtained with 60 load steps. $P_4 = \left(\frac{29}{14}a, \frac{13}{4}a, \frac{43}{70}a\right)$; $a = 50$ mm.
VIII. Conclusions

A reduced order technique for the aeroelastic analysis of planar and challenging non-planar configurations (joined wings) with the presence of important geometrical nonlinearities has been presented. The method consists in the writing of the linear system of equations that is solved at each iteration of the Newton Raphson procedure in modal generalized coordinates. A small number of low-frequency structural tangent modes is used to create the matrix that transforms the vector of physical nodal displacements in a vector of modal generalized coordinates. The structural tangent modes are reconstructed as a function of the load step (directly related to the aerodynamic pressure) via Proper Orthogonal Decomposition (POD) procedure. The POD modes are constant and independent of the dynamic pressure whereas the POD coefficients are a function of the aerodynamic pressure.

The study confirmed the difficulties associated with the modal reduction techniques when problems with important geometric nonlinearities are considered. In particular, it has been shown that joined wings, which have significant in-plane forces in the joint area, are very well represented by a relatively small number of low-frequency modes up to the aerodynamic speed $V_{POD}$ which is the limit of validity of the POD analysis. If higher speeds are analyzed ($V > V_{POD}$) the approximation is less accurate. This is not the case of less challenging configurations such as delta wings: an extrapolation of about 20% of the aerodynamic pressure ($V = 1.1V_{POD}$) is still well represented by the set of structural tangent modes reconstructed via POD analysis.

Two different extrapolation procedures have been proposed. The first one is based on a fitting process of the absolute values of the POD coefficients with respect to the dynamic pressure. The second procedure does not extrapolate the POD coefficients: for speeds larger than $V_{POD}$ the coefficients are assumed constant and equal to the coefficients evaluated at $V = V_{POD}$. The 2 methods of extrapolation had different numerical performances for the planar and non-planar wing configuration. This is a clearly problem-dependent feature.

The drawback of this reduced order technique for geometrically nonlinear aeroelastic analyses of wings is in the initial computational cost associated with the POD analysis that allows the reconstruction of the structural tangent modes. However, it has to be pointed out that alternative methods have similar issue: before the actual nonlinear analysis a large set of static nonlinear analyses is required to determine the unknown quadratic and cubic modal stiffnesses required to reduce the order of the system. Possible future investigation on this subject could be the following:

- Reduced Order Nonlinear aeroelastic analysis of joined wings with alternative available methods.
- Comparison between the computational cost of the initial procedures required by the proposed approach (initial POD analysis to find the POD modes and coefficients adopted to reconstruct the structural tangent modes) and the cost required by alternative methods that require the solution of a large number of static nonlinear analyses for the determination of the quadratic and cubic modal stiffness coefficients.
- Modification of the proposed procedure with selection of a different parameter (currently the load step $\lambda$ is used) for the reconstruction of the structural tangent modes. A selection of a parameter directly related to the actual internal stress level would probably be more physically representative and could lead to a better computational performances when the extrapolation is performed.
- Inclusion of higher order modes and reconstruction of their variability via Proper Orthogonal Decomposition.

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References