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# $\infty^6$ Mixed plate theories based on the Generalized Unified Formulation. Part I: Governing equations

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A R T I C L E  I N F O

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A B S T R A C T

The generalized unified formulation (GUF) is a formal technique which was introduced in the framework of displacement-based theories. GUF is extended for the first time to the case in which a mixed variational statement (Reissner's mixed variational theorem) is used. Each of the displacement variables and out-of-plane stresses is independently treated and different orders of expansions for the different unknowns can be chosen. Since infinite combinations can be freely chosen for the displacements $u_x, u_y, u_z$ and for the stresses $\sigma_{xz}, \sigma_{yz}, \sigma_{zz}$, the generalized unified formulation allows the user to write, with a single formal theory, $\infty^6$ theories which can be successfully implemented in a *single* FEM code. In addition, this formulation allows the user to treat each unknown independently and, therefore, different numerical approaches can be used in the FEM codes based on this generalized unified formulation. All the theories are originated from 13 independent fundamental nuclei (kernels of the generalized unified formulation) which are *formally invariant* and the layerwise mixed theories (analyzed in *Part II*), mixed higher order shear deformation theories (analyzed in *Part III*) and advanced mixed zig-zag models (analyzed in *Part IV*) can be studied without extra implementations or theoretical developments. Numerical performances and convergence properties of a very large amount of new mixed theories are discussed (*Part V*) with particular focus on the effects of the orders of expansion in the thickness direction of the displacements and modeled stresses. Multilayered composite plates will be analyzed. Different mixed variational statements could be used and the formulation could be easily adopted for multifield problems such as thermoelastic applications and multilayered plates embedding piezo-layers.



## 1. Introduction

### *1.1. What are the advantages of the generalized unified formulation (GUF) for the generation of an infinite number of theories for multilayered composite plates*

Classical plate theory (CPT), also known as Kirchoff theory [1], has the advantage of being simple and reliable for thin plates. However, if there is strong anisotropy of the mechanic properties, or if the composite plate is relatively thick, other advanced models such as first order shear deformation theory (FSDT) are required [2–4]. But even these theories are not sufficient if local effects are important or accuracy in the calculation of transverse stresses is sought. Therefore, more advanced plate theories have been developed to include zig-zag effects (among them see [5–11]). Higher order shear deformation theories (HSDT) have also been used [12–15], giving the possibility to increase the accuracy of numerical evaluations for moderately thick plates. A capability to freely change the orders used for the expansions of the single variables is then particularly useful. In fact, with the above mentioned axiomatic models, a new combination of orders implies a different theoretical formulation and different FE matrices. Also, it is desirable to have the freedom to numerically experiment the type of expansion for better performances with the geometry, boundary conditions and loads applied to the structure. If each "numerical experiment" requires a new theoretical development and a writing of a new code, it is practically impossible or too costly to actually perform such numerical approach. Also, in some problems a higher order theory (such as in the case of thick plates) is required and in others a low order theory (such as the CPT) is sufficient. *All these needs can be met with the use of the generalized unified formulation.* The generalized unified formulation is an extension of Carrera's unified formulation (CUF) [16–31]. When GUF is used the FEM matrices are derived from fundamental nuclei with dimension $1 \times 1$. Such fundamental nuclei (or kernels of the generalized unified formulation) are invariant with respect to the orders used in the expansions of the variables in the thickness direction. GUF is also indicated for *multifield* problems. In the multifield problems the final matrices will be in *any case* obtained by expanding $1 \times 1$ kernels as in the "pure" mechanical case. Another important feature of GUF is that each unknown, whether it is a displacement or a stress, is treated independently from the other

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unknowns. This is particularly appealing in the FEM discretization because it allows *different numerical treatment for each of the variables*. This property of GUF could lead to the derivation of new numerical techniques. Another property of GUF is that different orders of expansion of the variables can be used. This is important in view of the creation of a model which is capable to enhance the importance of some variables with respect to others (for example, for thin plates the displacement $u_z$ needs less terms in the thickness direction than the in-plane displacements $u_x$ and $u_y$, but for thick or very thick plates this is no longer true). GUF can also be used for an extensive analysis of *thickness locking problems* (see [32]).

This paper deals with theories based on Reissner's mixed variational theorem (RMVT) (see Refs. [33,34]). Infinite combinations can be chosen for the expansion of a single displacement or transverse stress. Considering then that the modeled fields are *six* (displacements $u_x, u_y, u_z$ and stresses $\sigma_{zx}$, $\sigma_{zy}$, $\sigma_{zz}$) it is deduced that $\infty^6$ different theories can be generated with the same formulation and with the same code.

### 1.2. What are the new contributions of this work

The generalized unified formulation was introduced for the first time in the particular case [35] of single layer isotropic plates. It was stated (but not demonstrated) that the formulation could be extended to the case of multilayered composite structures and could include *layerwise*, *equivalent single layer* and *zig-zag theories*. It was also stated that the formulation could be applied to the case of mixed variational statements. In a second paper [36] GUF was applied to a more general case of orthotropic materials and single layer plates. The stability of the solution (i.e., the displacements) was demonstrated in both Refs. [35,36]. In particular, it was shown that even in the case of very different orders used for the displacements $u_x, u_y$ and $u_z$ the solutions did not present oscillations and were "stable". GUF was introduced for the case of displacement-based theories (i.e., the principle of virtual displacements (PVD) was the variational statement). The previous conclusions were valid for the theories obtained using PVD.

The present work is divided into five parts (see [37–40]) which will answer the following key-questions:

- *Question # 1*
  How is the generalized unified formulation extended to the mixed cases (i.e., the used variational statement *is not* the principle of virtual displacements but a mixed one)?

This problem is considered in *Part I* of the present work, where the governing equations valid for the mixed case in which the variational statement is represented by Reissner's mixed variational theorem are introduced. The expressions of the $1 \times 1$ kernels of the generalized unified formulation are introduced. These kernels are invariant with respect to the orders used in the expansion of the displacements and transverse stresses along the thickness. The kernels are also invariant with respect to the type of theory: layer wise theories, higher order shear deformation theories and zig-zag theories, which all *have the same kernels*. All the FE matrices (if a FEM approach is used) are obtained from these kernels. Different variational statements produce different kernels. For example, in the PVD-based theories [35] six kernels were used. But in the case of mixed theories based on Reissener's variational theorem 13 kernels are required.

- *Question # 2*
  How is the generalized unified formulation extended to the case of multilayered structures?

This question is answered in *Part I* and *Part II* (see Ref. [37]) of the present work. Several examples explain how the pressures are applied within the formalism of GUF. Moreover, the expansion of the kernels introduced in *Part I* and the consequent generation of the layer matrices and their assembling at multilayer level are shown. The calculation of the in-plane stresses with the classical form of Hooke's law and mixed form of Hooke's law is discussed. Finally, the expanded matrices are explicitly reported for a particular case to help the readers reproduce the results of this work. Each layer can be either isotropic or orthotropic (composites). The layers can be arranged in any combination and no symmetry with respect to the middle plane is required. *Part III* (Ref. [38]) and *Part IV* (Ref. [39]) discuss equivalent single layer theories for multilayered structures.

- *Question # 3*
  How is the generalized unified formulation extended to the case of layerwise theories?

This question is also answered in *Part II* (Ref. [37]) of the present work. The imposition of the interlaminar continuity of the displacements (compatibility condition) and transverse stresses (equilibrium condition) requires a class of functions used in the thickness expansion of the variables with some peculiar properties. It is demonstrated (*Part II*) that a particular combination of Legendre polynomials satisfies these goals. Most of the derivations reported in *Part II* could be successfully applied to the case in which PVD-based layerwise theories are considered.

- *Question # 4*
  How is the generalized unified formulation applied to the case of equivalent single layer theories such as mixed higher order shear deformation theories?

This issue is discussed in *Part III* (Ref. [38]). In particular, it is demonstrated that the kernels obtained in *Part I* can be formally used for the generation of the layer matrices of these higher order shear deformation theories. The transverse stresses $\sigma_{zx}, \sigma_{zy}$ and $\sigma_{zz}$ can be kept as unknowns. If so, the *quasi-layerwise Reissner's mixed variational theorem-based higher order shear deformation theories (QLRHSDT)* are obtained. These theories are discussed in *Part III*. If the transverse stresses *are not* kept as unknowns and the static condensation technique is performed, *Reissner's mixed variational theorem-based higher order shear deformation theories (RHSDT)* are obtained. How the theories are created with the same kernels and formalism of GUF is explained with numerous examples in *Part III*. The equilibrium and compatibility conditions are enforced *a priori*.

- *Question # 5*
  How is the generalized unified formulation extended to the case of equivalent single layer theories which include the zig-zag effects?

The inclusion of Murakami's zig-zag function is discussed in the case of theories based on GUF. This discussion is presented in *Part IV* (Ref. [39]). In particular, *Reissner's mixed variational theorem-based zig-zag theories (RZZT)* and *quasi-layerwise Reissner's mixed variational theorem-based zig-zag theories (QLRZZT)* are introduced and analyzed in detail. How these theories are again obtained from the same kernels of the generalized unified formulation (presented in *Part I*) is discussed with numerous examples. The equilibrium and compatibility conditions are enforced *a priori*, as in the cases analyzed in *Part II* and *Part III*.

- *Question # 6*
  In the case in which the used variational statement is Reissner's mixed variational theorem, will the unknown fields represented by the displacements $u_x, u_y$ and $u_z$ and the stresses $\sigma_{zx}, \sigma_{zy}$ and $\sigma_{zz}$ be numerically "stable" without oscillations?

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This important question is answered in *Part V* (Ref. [40]). It is demonstrated that the relative orders used for the displacements and transverse stresses are key factors that determine the numerical stability of the solution represented by the displacement and stress fields. In some cases the orders are such that the displacements show numerical oscillations. In some other cases the out-of-plane stresses calculated *a priori* present oscillations. Several classes of theories are studied and the stable and unstable cases are discussed in detail. New benchmarks are presented and a large amount of new theories are presented for the first time in the literature.

- *Question # 7*
  In case the oscillations are possible, under which conditions do they arise? How can the oscillations be eliminated or reduced? Do layerwise theories behave differently with respect to the equivalent single layer theories with or without zig-zag effects?

These issues are discussed in *Part V* (Ref. [40]). "Practical" recipes to avoid numerical difficulties are given with examples. New test cases are proposed to asses a very large amount of new layerwise, higher order shear deformation and zig-zag theories. It is demonstrated that the behavior of the equivalent single layer theories is different than the behavior shown by the layerwise theories. It is also presented that many key numerical performances of equivalent single layer models can be explained by considering the class of theories presented for the layerwise cases. Therefore, it is demonstrated that the layerwise theories are fundamental in the study of equivalent single layer models with or without the inclusion of zig-zag effects. Past studies [31] about poor convergence of the equivalent single layer theories, as far as the *a priori* out-of-plane stresses are concerned, are reinterpreted using the finding of this work. Carrera's findings are integrated with the present study and a new light is shed on the issue of the calculation of out-of-plane shear and normal stresses with Reissner's mixed variational theorem.

- *Question # 8*
  In the literature it was demonstrated that in equivalent single layer theories based on Reissner's mixed variational theorem it was *always* convenient to add Murakami's zig-zag function to improve the accuracy of the results. Is this true in *all* cases when the orders of the displacements and stresses are let free to vary?

This question is again answered in *Part V* (Ref. [40]). It is demonstrated that it is *not always* convenient to add the zig-zag function. This is a less intuitive result and can be demonstrated considering the numerical performances of the mixed theories when the orders of displacements are changed independently from the orders used for the stresses. A zig-zag theory could be seen as a higher order shear deformation theory with the inclusion of a zig-zag function. The initial orders used in the starting higher order shear deformation theory that is being "improved" (with the addition of the zig-zag function) affect the answer to this question.

## 2. Classification of the theories

The main feature of the generalized unified formulation is that the descriptions of layerwise theories, higher order shear deformation theories and zig-zag theories *do not show any formal differences*. So, with just one theoretical model an infinite number of different approaches can be considered. For example, in the case of moderately thick plates a higher order theory could be sufficient but for thick plates layerwise models may be required. With GUF the two approaches are formally identical because the kernels are invariant with respect to the theory.

In the present work the concepts of *type of theory* and *class of theories* are introduced. The following types of theories are discussed:

- Reissner's mixed variational theorem-based higher order shear deformation theories (RHSDT)
  These theories are equivalent single layer models because the displacement field is unique and independent of the number of layers. These theories will be discussed in *Part III*.
- Quasi-layerwise Reissner's mixed variational theorem-based higher order shear deformation theories (QLRHSDT)
  These theories are equivalent single layer models for the displacement fields but the out-of-plane stresses have a layerwise description. The difference between RHSDT and QLRHSDT is that in the first case the static condensation technique (SCT) is applied and the stresses are eliminated and calculated a posteriori. These theories will be discussed in *Part III*.
- Reissner's mixed variational theorem-based higher order shear deformation theories with zig-zag effects included (RHSDTZ)
  These theories are equivalent single layer models and the so called zig-zag form of the displacements is taken into account by using Murakami's zig-zag function (MZZF). The static condensation technique *is* performed and the stresses $\sigma_{zx}, \sigma_{zy}$ and $\sigma_{zz}$ are calculated a posteriori (but their continuity is imposed *a priori*). These theories will be discussed in *Part IV*.
- Quasi-layerwise Reissner's mixed variational theorem-based higher order shear deformation theories with zig-zag effects included (QLRHSDTZ)
  These theories are equivalent single layer models and the so called zig-zag form of the displacements is taken into account by using Murakami's zig-zag function (MZZF). The static condensation technique *is not* performed and the stresses $\sigma_{zx}, \sigma_{zy}$ and $\sigma_{zz}$ are calculated *a priori*. These theories will be discussed in *Part IV*.
- Layerwise Reissner's mixed variational theorem-based theories (LRT)
  These theories are the most accurate ones because all the displacements and out-of-plane stresses have a layerwise description. These models are necessary when local effects need to be described. The price is of course (in FEM applications) in higher computational time. These theories will be discussed in *Part II*.

An infinite number of theories which have a particular logic in the selection of the used orders of expansion is defined as *class of theories*. For example, the infinite layerwise theories which have the displacements $u_x, u_y$ and $u_z$ expanded along the thickness with a polynomial of order $N$ and which have the stresses $\sigma_{zx}, \sigma_{zy}$ and $\sigma_{zz}$ expanded along the thickness with a polynomial of order $N+3$ are a class of theories. The infinite theories which have the in-plane displacements $u_x$ and $u_y$ expanded along the thickness with order $N$, the out of plane displacement expanded along the thickness with order $N-1$ and the stresses expanded along the thickness with order $M$ are another class of theories.

The concept of *class of theories* will be discussed in detail in *Part V* and will be used to address the numerical instabilities that may arise in some cases.

## 3. The generalized unified formulation for multilayered composite plates

Both layerwise and equivalent single layer models are axiomatic approaches. That is, the unknowns are expanded along the thickness by using *a chosen* series of functions.

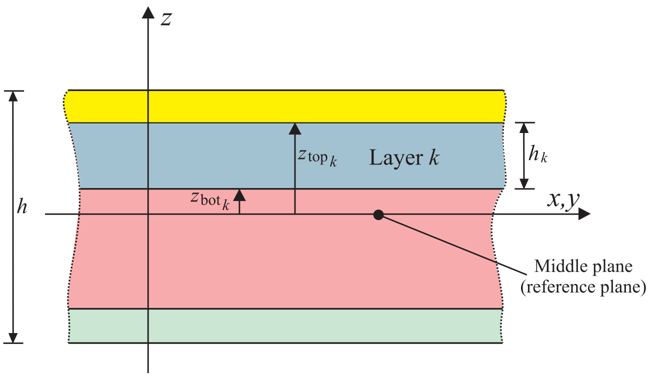


**Fig. 1.** Multilayered plate: notations and definitions.

When the principal of virtual displacements is used, the unknowns are the displacements. When Reissner's mixed variational theorem is used the unknowns are the displacements *and* the out-of-plane stresses. This variational statement has then the advantage that the compatibility of the displacements and the equilibrium between two adjacent layers can be "naturally" satisfied. This is the main reason why RMVT can be considered a powerful tool for the analysis of multilayered plates.

The generalized unified formulation is introduced here considering a generic layer $k$ of a multilayered plate structure. This is the most general approach and the equivalent single layer theories, which consider the displacement unknowns to be layer-independent, can be derived from this formulation with some simple formal techniques as will be demonstrated in *Part III* and *Part IV*. To begin with, consider a theory denoted as Theory I, in which the displacement in $x$ direction $u_x^k$ has four degrees of freedom.[1] This means that for displacement $u_x^k$ we have four unknowns. Each unknown multiplies *a known function* of the thickness coordinate $z$. Where the origin of the coordinate $z$ is measured is not important. However, from a practical point of view it is convenient to assume that the *middle plane* of the plate is also the plane with $z = 0$. For layer $k$ is then $z_{\text{bot}_k} \leqslant z \leqslant z_{\text{top}_k}$. $z_{\text{bot}_k}$ is the global coordinate $z$ of the bottom surface of layer $k$ and $z_{\text{top}_k}$ is the global coordinate $z$ of the top surface of layer $k$ (see Fig. 1). $h_k = z_{\text{top}_k} - z_{\text{bot}_k}$ is the thickness of layer $k$ and $h$ is the thickness of the plate.

The $x$ component of the displacement vector of layer $k$ is indicated with $u_x^k$. In the case of Theory I, $u_x^k$ is expressed as follows:

$$u_x^k(x,y,z) = \overbrace{f_1^k(z)}^{\text{known}} \cdot \overbrace{u_{x_1}^k(x,y)}^{\text{unknown 1}} + \overbrace{f_2^k(z)}^{\text{known}} \cdot \overbrace{u_{x_2}^k(x,y)}^{\text{unknown 2}} + \overbrace{f_3^k(z)}^{\text{known}} \cdot \overbrace{u_{x_3}^k(x,y)}^{\text{unknown 3}} + \overbrace{f_4^k(z)}^{\text{known}} \cdot \overbrace{u_{x_4}^k(x,y)}^{\text{unknown 4}} \qquad z_{\text{bot}_k} \leqslant z \leqslant z_{\text{top}_k} \tag{1}$$

The functions $f_1^k(z), f_2^k(z), f_3^k(z)$ and $f_4^k(z)$ are *known* functions (axiomatic approach). These functions could be, for example, a series of trigonometric functions of the thickness coordinate $z$. Polynomials (or even better orthogonal polynomials) could be selected. In the most general case each layer has different functions. For example, $f_1^k(z) \neq f_1^{k+1}(z)$. The next *formal step* is to modify the notation.

The following functions are *defined*:

$$\begin{aligned} {}^xF_{\text{t}}^k(z) &= f_1^k(z) \qquad {}^xF_2^k(z) = f_2^k(z) \\ {}^xF_3^k(z) &= f_3^k(z) \qquad {}^xF_{\text{b}}^k(z) = f_4^k(z) \end{aligned} \tag{2}$$

[1] In FEM applications the number of degrees of freedom depends also on the mesh. Here we consider only the degrees of freedom related to the expansions in the thickness direction.

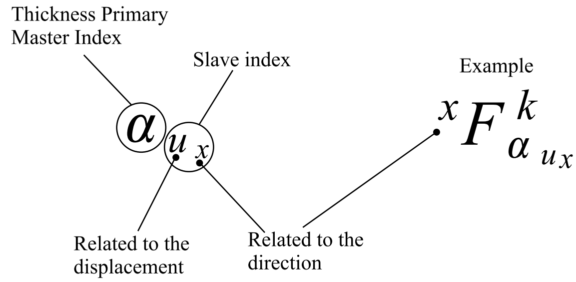


**Fig. 2.** generalized unified formulation. Master and slave indices.

The logic behind these definitions is the following:

- The *first* function $f_1^k(z)$ is defined as ${}^xF_{\text{t}}^k$. Notice the superscript $x$. It was added to clarify that the displacement in $x$ direction, $u_x^k$, is under investigation. The subscript $t$ now seems without any utility. However, later it will identify the quantities at the "top" of the plate and, therefore, will be useful in the assembling of the stiffness matrices in the thickness direction. This aspect will be discussed in *Part II*.
- The *last* function $f_4^k(z)$ is defined as ${}^xF_{\text{b}}^k$. Notice again the superscript $x$. The subscript $b$ means "bottom" and, again, its utility will be clear when the matrices are assembled.
- The intermediate functions $f_2^k(z)$ and $f_3^k(z)$ are defined simply as ${}^xF_2^k$ and ${}^xF_3^k$.

To be consistent with the definitions of equation 2, the following unknown quantities are *defined*:

$$u_{x_{\text{t}}}^k(x,y) = u_{x_1}^k(x,y) \qquad u_{x_{\text{b}}}^k(x,y) = u_{x_4}^k(x,y) \tag{3}$$

Using the definitions reported in Eqs. (2) and (3), Eq. (1) can be rewritten as

$$u_x^k(x,y,z) = \overbrace{{}^xF_{\text{t}}^k(z)}^{\text{known}} \cdot \overbrace{u_{x_{\text{t}}}^k(x,y)}^{\text{unknown 1}} + \overbrace{{}^xF_2^k(z)}^{\text{known}} \cdot \overbrace{u_{x_2}^k(x,y)}^{\text{unknown 2}} + \overbrace{{}^xF_3^k(z)}^{\text{known}} \cdot \overbrace{u_{x_3}^k(x,y)}^{\text{unknown 3}} + \overbrace{{}^xF_{\text{b}}^k(z)}^{\text{known}} \cdot \overbrace{u_{x_{\text{b}}}^k(x,y)}^{\text{unknown 4}} \qquad z_{\text{bot}_k} \leqslant z \leqslant z_{\text{top}_k} \tag{4}$$

It is supposed that each function of $z$ is a polynomial. The order of the expansion is then 3 and indicated as $N_{u_x}^k$. Each layer has in general a different order. Thus, in general $N_{u_x}^k \neq N_{u_x}^{k+1}$. If the functions of $z$ are not polynomials (for example, this is the case if trigonometric functions are used) then $N_{u_x}^k$ is just a parameter related to the number of terms or degrees of freedom used to describe the displacement $u_x^k$ in the thickness direction. This concept will be clear later. The expression representing the displacement $u_x^k$ can be put in a compact form by using the generalized unified formulation [35] as follows:

$$u_x^k(x,y,z) = {}^xF_{\alpha_{u_x}}^k(z) \cdot u_{x\alpha_{u_x}}^k(x,y) \quad \alpha_{u_x} = t, l, b; \quad l = 2, \ldots, N_{u_x}^k \tag{5}$$

where, in the example, $N_{u_x}^k = 3$. The thickness primary master index $\alpha$ has the subscript $u_x$. This subscript from now on will be called *slave index*. It is introduced to show that the displacement $u_x$ is considered. Fig. 2 explains these definitions.

Consider another example. Suppose that the displacement $u_x^k$ of a particular theory is expressed with 3 degrees of freedom. In that case it is possible to write:

$$u_x^k(x,y,z) = \overbrace{f_1^k(z)}^{\text{known}} \cdot \overbrace{u_{x_1}^k(x,y)}^{\text{unknown 1}} + \overbrace{f_2^k(z)}^{\text{known}} \cdot \overbrace{u_{x_2}^k(x,y)}^{\text{unknown 2}} + \overbrace{f_3^k(z)}^{\text{known}} \cdot \overbrace{u_{x_3}^k(x,y)}^{\text{unknown 3}} \tag{6}$$

By adopting the definitions earlier used for the case of 4 degrees of freedom it is possible to rewrite Eq. 6 in the following equivalent form:

$$u_x^k(x,y,z) = \overbrace{{}^xF_t^k(z)}^{\text{known}} \cdot \overbrace{u_{x_t}^k(x,y)}^{\text{unknown 1}} + \overbrace{F_2^k(z)}^{\text{known}} \cdot \overbrace{u_{x_2}^k(x,y)}^{\text{unknown 2}} + \overbrace{{}^xF_b^k(z)}^{\text{known}} \cdot \overbrace{u_{x_b}^k(x,y)}^{\text{unknown 3}} \tag{7}$$

which can be put again in the form shown in Eq. 5 with $N_{u_x}^k = 2$. It can then be deduced that:

- $N_{u_x}^k$ is $\text{DOF}_{u_x}^k - 1$, where $\text{DOF}_{u_x}^k$ is the number of degrees of freedom (at layer level) used for the displacement $u_x^k$. In the case of zig-zag theories it will be shown that $N_{u_x}^k = \text{DOF}_{u_x}^k - 2$ because one degree of freedom is used for the zig-zag function. However, for now this aspect will not be considered and will be analyzed in detail in *Part IV*.
- The minimum number of degrees of freedom is 2. This is a choice used to facilitate the assembling in the thickness direction. In fact, the "top" and "bottom" terms will be always present. In the case in which $\text{DOF}_{u_x}^k = 2$ the generalized unified formulation is simply

$$u_x^k(x,y,z) = {}^xF_{\alpha_{u_x}}^k(z) \cdot u_{x\alpha_{u_x}}^k(x,y) \quad \alpha_{u_x} = t, b \tag{8}$$

  Notice that the "$l$" term of Eq. (5) is not present.
- An *infinite* number of theories can be included in Eq. (5). It is in fact sufficient to change the value of $N_{u_x}^k$. It should be observed that *formally* there is no difference between two distinct theories (obtained by changing $N_{u_x}^k$). It is introduced the terminology that $\infty^1$ theories can be represented by Eq. (5).

The other displacements $u_y^k$ and $u_z^k$ can be treated in a similar fashion. The stresses $\sigma_{zx}^k, \sigma_{zy}^k$ and $\sigma_{zz}^k$ can be represented by the generalized unified formulation. To do so it is sufficient to *define*

$$s_x^k = \sigma_{zx}^k \qquad s_y^k = \sigma_{zy}^k \qquad s_z^k = \sigma_{zz}^k \tag{9}$$

and treat, from a formal point of view, $s_x^k, s_y^k$ and $s_z^k$ as displacements. Considering what has been said, the generalized unified formulation for all the *displacements* and *out-of-plane stresses* is the following:

$$\begin{aligned}
u_x^k &= {}^xF_t u_{x_t}^k + {}^xF_l u_{x_l}^k + {}^xF_b u_{x_b}^k = {}^xF_{\alpha_{u_x}} u_{x\alpha_{u_x}}^k \\
&\alpha_{u_x} = t, l, b; \quad l = 2, \ldots, N_{u_x} \\
u_y^k &= {}^yF_t u_{y_t}^k + {}^yF_m u_{y_m}^k + {}^yF_b u_{y_b}^k = {}^yF_{\alpha_{u_y}} u_{y\alpha_{u_y}}^k \\
&\alpha_{u_y} = t, m, b; \quad m = 2, \ldots, N_{u_y} \\
u_z^k &= {}^zF_t u_{z_t}^k + {}^zF_n u_{z_n}^k + {}^zF_b u_{z_b}^k = {}^zF_{\alpha_{u_z}} u_{z\alpha_{u_z}}^k \\
&\alpha_{u_z} = t, n, b; \quad n = 2, \ldots, N_{u_z} \\
s_x^k &= {}^x\mathcal{F}_t s_{x_t}^k + {}^x\mathcal{F}_p s_{x_p}^k + {}^x\mathcal{F}_b s_{x_b}^k = {}^x\mathcal{F}_{\alpha_{s_x}} s_{x\alpha_{s_x}}^k \\
&\alpha_{s_x} = t, p, b; \quad p = 2, \ldots, N_{s_x} \\
s_y^k &= {}^y\mathcal{F}_t s_{y_t}^k + {}^y\mathcal{F}_q s_{y_q}^k + {}^y\mathcal{F}_b s_{y_b}^k = {}^y\mathcal{F}_{\alpha_{s_y}} s_{y\alpha_{s_y}}^k \\
&\alpha_{s_y} = t, q, b; \; q = 2, \ldots, N_{s_y} \\
s_z^k &= {}^z\mathcal{F}_t s_{z_t}^k + {}^z\mathcal{F}_r s_{z_r}^k + {}^z\mathcal{F}_b s_{z_b}^k = {}^z\mathcal{F}_{\alpha_{s_z}} s_{z\alpha_{s_z}}^k \\
&\alpha_{s_z} = t, r, b; \quad r = 2, \ldots, N_{s_z}
\end{aligned} \tag{10}$$

For the out-of-plane stresses $s_x^k, s_y^k$ and $s_z^k$ the functions of the thickness coordinate are indicated with the symbol $\mathcal{F}$ instead of the symbol $F$ to distinguish the case in which stresses are considered from the case in which displacements are considered. Also, for the stresses, the slave indices $s_x, s_y$ and $s_z$ are used instead of $u_x, u_y$ and $u_z$. The formalism is very similar in both the displacements and out-of-plane stresses cases. In Eq. (10), for simplicity it is assumed that the type of functions is the same for each layer and that the same number of terms is used for each layer. This assumption will make it possible to adopt the same generalized unified formulation for all types of theories, and layerwise and equivalent single layer theories will *not* show formal differences. This concept means, for example, that if displacement $u_y$ is approximated with four terms in a particular layer $k$ then it will be approximated with four terms in *all* layers of the multilayered structure.

Each variable can be expanded in $\infty^1$ combinations. In fact, it is sufficient to change the number of terms used for each variable. Since there are six variables (the displacements and transverse stresses), it is concluded that Eq. (10) includes $\infty^6$ different theories. For example, it is possible to have a plate theory with

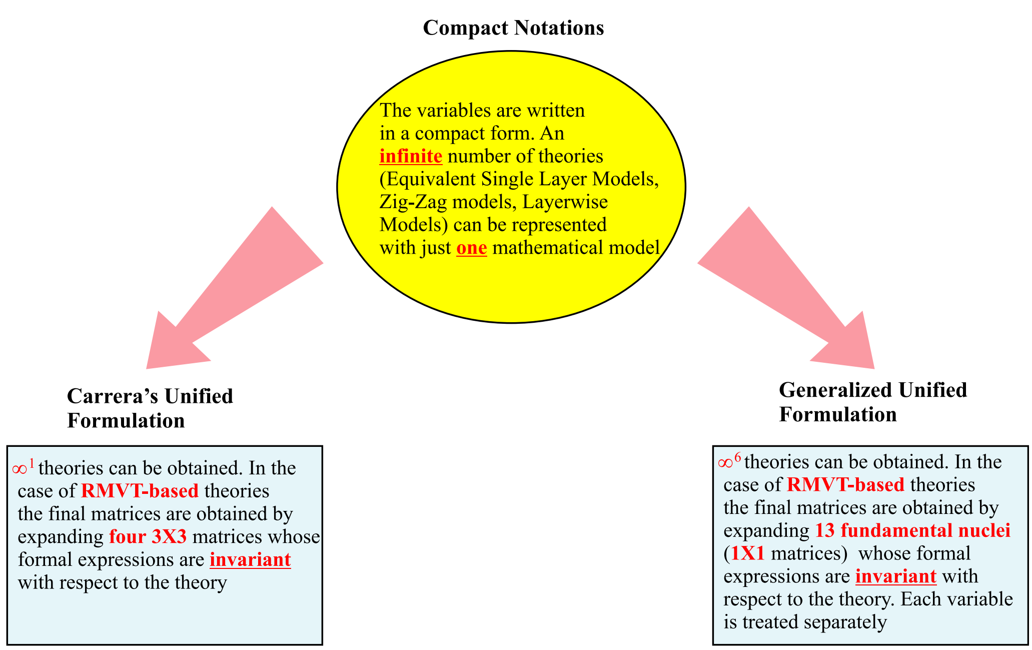


**Fig. 3.** Concepts of compact notations in the case of RMVT-based theories. Carrera's unified formulation and generalized unified formulation.

cubic expansion along the thickness of the in-plane displacements and out-of-plane shear stresses and parabolic expansion along the thickness of the displacement $u_z^k$ and transverse stress $\sigma_{zz}^k = s_z^k$. The user can freely decide the theory, and without formulating a new theoretical development and modifying the code $\infty^6$ theories can be tested and assessed. From Eq. (10) it is also deduced that every variable can be independently treated, and this is particularly useful in the implementations of new numerical techniques. Also, the user can change theory considering the needs. For example, if a thin plate is analyzed, the order of out-of-plane displacement $u_z$ can be small. On the contrary, to get the convergence in the case of thick plates the orders used for the different variables have to be increased. All this freedom is the power of the generalized unified formulation: in one formalism $\infty^6$ theories are included. The generalized unified formulation is a generalization of Carrera's unified formulation [31]. The main difference between the two compact notations are summarized in Fig. 3, where the case of Reissner's mixed variational theorem is considered. if the variational statement is changed then the number of fundamental nuclei required to generate the theories is different.

## 4. Classical form of Hooke's law (CFHL)

To apply RMVT the mixed form of Hooke's law (MFHL), which is obtained from the classical form of Hooke's law (CFHL), is required. For this reason, the classical form of Hooke's law (CFHL), which relates the stresses to the strains is discussed. Its explicit form for *orthotropic materials* is the following:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \\ \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \widetilde{C}_{11} & \widetilde{C}_{12} & \widetilde{C}_{16} & 0 & 0 & \widetilde{C}_{13} \\ \widetilde{C}_{12} & \widetilde{C}_{22} & \widetilde{C}_{26} & 0 & 0 & \widetilde{C}_{23} \\ \widetilde{C}_{16} & \widetilde{C}_{26} & \widetilde{C}_{66} & 0 & 0 & \widetilde{C}_{36} \\ 0 & 0 & 0 & \widetilde{C}_{55} & \widetilde{C}_{45} & 0 \\ 0 & 0 & 0 & \widetilde{C}_{45} & \widetilde{C}_{44} & 0 \\ \widetilde{C}_{13} & \widetilde{C}_{23} & \widetilde{C}_{36} & 0 & 0 & \widetilde{C}_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \\ \varepsilon_{zz} \end{bmatrix} \qquad (11)$$

where

$$\begin{aligned}
\widetilde{C}_{11} &= C^4\overline{C}_{11} + 2S^2C^2\overline{C}_{12} + S^4\overline{C}_{22} + 4S^2C^2\overline{C}_{66} \\
\widetilde{C}_{12} &= S^2C^2\overline{C}_{11} + C^4\overline{C}_{12} + S^4\overline{C}_{12} + S^2C^2\overline{C}_{22} \\
&\quad - 4S^2C^2\overline{C}_{66} \\
\widetilde{C}_{16} &= SC^3\overline{C}_{11} + S^3C\overline{C}_{12} - SC^3\overline{C}_{12} - S^3C\overline{C}_{22} \\
&\quad - 2SC^3\overline{C}_{66} + 2S^3C\overline{C}_{66} \\
\widetilde{C}_{13} &= C^2\overline{C}_{13} + S^2\overline{C}_{23} \\
\widetilde{C}_{22} &= S^4\overline{C}_{11} + 2S^2C^2\overline{C}_{12} + C^4\overline{C}_{22} + 4S^2C^2\overline{C}_{66} \\
\widetilde{C}_{26} &= S^3C\overline{C}_{11} + SC^3\overline{C}_{12} - S^3C\overline{C}_{12} - SC^3\overline{C}_{22} \\
&\quad + 2SC^3\overline{C}_{66} - 2S^3C\overline{C}_{66} \\
\widetilde{C}_{23} &= S^2\overline{C}_{13} + C^2\overline{C}_{23} \\
\widetilde{C}_{66} &= S^2C^2\overline{C}_{11} - 2S^2C^2\overline{C}_{12} + S^2C^2\overline{C}_{22} \\
&\quad + \overline{C}_{66}C^4 - 2S^2C^2\overline{C}_{66} + \overline{C}_{66}S^4 \\
\widetilde{C}_{36} &= SC\overline{C}_{13} - SC\overline{C}_{23} \\
\widetilde{C}_{55} &= C^2\overline{C}_{55} + S^2\overline{C}_{44} \\
\widetilde{C}_{45} &= SC\overline{C}_{55} - SC\overline{C}_{44} \\
\widetilde{C}_{44} &= S^2\overline{C}_{55} + C^2\overline{C}_{44} \\
\widetilde{C}_{33} &= \overline{C}_{33}
\end{aligned} \qquad (12)$$

The definitions $C = \cos\vartheta$ and $S = \sin\vartheta$ were used, where $\vartheta$ is the rotation angle between the material coordinates and problem coordinates [41]. If $\vartheta = 0/90$ then $\widetilde{C}_{16} = \widetilde{C}_{26} = \widetilde{C}_{36} = \widetilde{C}_{45} = 0$. The coefficients of Hooke's law in material coordinates are

$$\begin{aligned}
&\overline{C}_{11} = \frac{1 - \upsilon_{23}\upsilon_{32}}{\Delta}E_{11}; \quad \overline{C}_{12} = \frac{\upsilon_{21} + \upsilon_{23}\upsilon_{31}}{\Delta}E_{11}; \\
&\overline{C}_{22} = \frac{1 - \upsilon_{13}\upsilon_{31}}{\Delta}E_{22} \\
&\overline{C}_{13} = \frac{\upsilon_{21}\upsilon_{32} + \upsilon_{31}}{\Delta}E_{11}; \quad \overline{C}_{23} = \frac{\upsilon_{32} + \upsilon_{12}\upsilon_{31}}{\Delta}E_{22}; \\
&\overline{C}_{33} = \frac{1 - \upsilon_{12}\upsilon_{21}}{\Delta}E_{33} \\
&\overline{C}_{44} = G_{23}; \quad \overline{C}_{55} = G_{13}; \quad \overline{C}_{66} = G_{12} \\
&\Delta = 1 - \upsilon_{23}\upsilon_{32} - \upsilon_{12}\upsilon_{21} - \upsilon_{13}\upsilon_{31} - 2\upsilon_{21}\upsilon_{32}\upsilon_{13} \\
&\upsilon_{32} = \frac{E_{33}}{E_{22}}\upsilon_{23}; \quad \upsilon_{21} = \frac{E_{22}}{E_{11}}\upsilon_{12}; \quad \upsilon_{31} = \frac{E_{33}}{E_{11}}\upsilon_{13}
\end{aligned} \qquad (13)$$

The *independent* parameters used in the definition of the material properties are nine: $\upsilon_{12}, \upsilon_{13}, \upsilon_{23}, G_{12}, G_{13}, G_{23}, E_{11}, E_{22}$ and $E_{33}$. In the case of isotropic materials only two parameters are needed: the Poisson's ratio and the elastic modulus.

## 5. Mixed form of Hooke's law (MFHL)

The classical form of Hooke's law can be written in a compact form as

$$\begin{bmatrix} \boldsymbol{\sigma}_p \\ \boldsymbol{\sigma}_n \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbf{C}}_{pp} & \widetilde{\mathbf{C}}_{pn} \\ \widetilde{\mathbf{C}}_{np} & \widetilde{\mathbf{C}}_{nn} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon}_p \\ \boldsymbol{\varepsilon}_n \end{bmatrix} \qquad (14)$$

where

$$\boldsymbol{\sigma}_p = [\sigma_{xx} \quad \sigma_{yy} \quad \sigma_{xy}]^T; \quad \boldsymbol{\sigma}_n = [\sigma_{xz} \quad \sigma_{yz} \quad \sigma_{zz}]^T \qquad (15)$$

$$\boldsymbol{\varepsilon}_p = [\varepsilon_{xx} \quad \varepsilon_{yy} \quad \gamma_{xy}]^T; \quad \boldsymbol{\varepsilon}_n = [\gamma_{xz} \quad \gamma_{yz} \quad \varepsilon_{zz}]^T \qquad (16)$$

The mixed form of Hooke's law is now derived. From the second relation of Eq. (14), it is possible to express $\boldsymbol{\varepsilon}_n$ as a function of $\boldsymbol{\sigma}_n$ and $\boldsymbol{\varepsilon}_p$:

$$\boldsymbol{\varepsilon}_n = (\widetilde{\mathbf{C}}_{nn})^{-1}\boldsymbol{\sigma}_n - (\widetilde{\mathbf{C}}_{nn})^{-1}\widetilde{\mathbf{C}}_{np}\boldsymbol{\varepsilon}_p \qquad (17)$$

Substituting into the expression of the in-plane stresses (first relation of Eq. (14)):

$$\boldsymbol{\sigma}_p = [\widetilde{\mathbf{C}}_{pp} - \widetilde{\mathbf{C}}_{pn}(\widetilde{\mathbf{C}}_{nn})^{-1}\widetilde{\mathbf{C}}_{np}]\boldsymbol{\varepsilon}_p + \widetilde{\mathbf{C}}_{pn}(\widetilde{\mathbf{C}}_{nn})^{-1}\boldsymbol{\sigma}_n \qquad (18)$$

Hence, MFHL is

$$\begin{bmatrix} \boldsymbol{\sigma}_p \\ \boldsymbol{\varepsilon}_n \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{pp} & \mathbf{C}_{pn} \\ \mathbf{C}_{np} & \mathbf{C}_{nn} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon}_p \\ \boldsymbol{\sigma}_n \end{bmatrix} \qquad (19)$$

where

$$\begin{aligned}
\mathbf{C}_{pp} &= \widetilde{\mathbf{C}}_{pp} - \widetilde{\mathbf{C}}_{pn}(\widetilde{\mathbf{C}}_{nn})^{-1}\widetilde{\mathbf{C}}_{np} \\
\mathbf{C}_{pn} &= \widetilde{\mathbf{C}}_{pn}(\widetilde{\mathbf{C}}_{nn})^{-1} \\
\mathbf{C}_{np} &= -(\widetilde{\mathbf{C}}_{nn})^{-1}\widetilde{\mathbf{C}}_{np} \\
\mathbf{C}_{nn} &= (\widetilde{\mathbf{C}}_{nn})^{-1}
\end{aligned} \qquad (20)$$

explicitly

$$\begin{aligned}
\mathbf{C}_{pp} &= \begin{bmatrix} \widetilde{C}_{11} - \frac{(\widetilde{C}_{13})^2}{\widetilde{C}_{33}} & \widetilde{C}_{12} - \frac{\widetilde{C}_{13}}{\widetilde{C}_{33}}\widetilde{C}_{23} & \widetilde{C}_{16} - \frac{\widetilde{C}_{13}}{\widetilde{C}_{33}}\widetilde{C}_{36} \\ \widetilde{C}_{12} - \frac{\widetilde{C}_{13}}{\widetilde{C}_{33}}\widetilde{C}_{23} & \widetilde{C}_{22} - \frac{(\widetilde{C}_{23})^2}{\widetilde{C}_{33}} & \widetilde{C}_{26} - \frac{\widetilde{C}_{23}}{\widetilde{C}_{33}}\widetilde{C}_{36} \\ \widetilde{C}_{16} - \frac{\widetilde{C}_{13}}{\widetilde{C}_{33}}\widetilde{C}_{36} & \widetilde{C}_{26} - \frac{\widetilde{C}_{23}}{\widetilde{C}_{33}}\widetilde{C}_{36} & \widetilde{C}_{66} - \frac{(\widetilde{C}_{36})^2}{\widetilde{C}_{33}} \end{bmatrix}; \\
\mathbf{C}_{pn} &= \begin{bmatrix} 0 & 0 & \frac{\widetilde{C}_{13}}{\widetilde{C}_{33}} \\ 0 & 0 & \frac{\widetilde{C}_{23}}{\widetilde{C}_{33}} \\ 0 & 0 & \frac{\widetilde{C}_{36}}{\widetilde{C}_{33}} \end{bmatrix}
\end{aligned} \qquad (21)$$

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$$\mathbf{C}_{np} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{\widetilde{C}_{13}}{\widetilde{C}_{33}} & -\frac{\widetilde{C}_{23}}{\widetilde{C}_{33}} & -\frac{\widetilde{C}_{36}}{\widetilde{C}_{33}} \end{bmatrix};$$

$$\mathbf{C}_{nn} = \begin{bmatrix} \frac{\widetilde{C}_{44}}{\widetilde{C}_{55}\widetilde{C}_{44}-(\widetilde{C}_{45})^2} & -\frac{\widetilde{C}_{45}}{\widetilde{C}_{55}\widetilde{C}_{44}-(\widetilde{C}_{45})^2} & 0 \\ -\frac{\widetilde{C}_{45}}{\widetilde{C}_{55}\widetilde{C}_{44}-(\widetilde{C}_{45})^2} & \frac{\widetilde{C}_{55}}{\widetilde{C}_{55}\widetilde{C}_{44}-(\widetilde{C}_{45})^2} & 0 \\ 0 & 0 & \frac{1}{\widetilde{C}_{33}} \end{bmatrix} \quad (22)$$

Formally, Hooke's law in the mixed case is

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \\ \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{16} & 0 & 0 & C_{13} \\ C_{12} & C_{22} & C_{26} & 0 & 0 & C_{23} \\ C_{16} & C_{26} & C_{66} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{55} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{44} & 0 \\ -C_{13} & -C_{23} & -C_{36} & 0 & 0 & C_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \\ \sigma_{zz} \end{bmatrix} \quad (23)$$

The expressions of the coefficients as functions of the CFHL coefficients can be deduced directly from Eqs. (23), (22), (21) and (19).

## 6. Reissner's mixed variational theorem (RMVT)

It is fundamental to have a tool which allows to model the transverse stresses and enforce the continuity through the thickness. Starting from the Hellinger–Reissner functional (see [42]), with a few transformations (partial Legendre transformation) it is not difficult to obtain the Reissner's mixed variational theorem ([33,34]):

$$\int_v [\delta\boldsymbol{\varepsilon}_{pG}^{\mathrm{T}}\boldsymbol{\sigma}_{pH} + \delta\boldsymbol{\varepsilon}_{nG}^{\mathrm{T}}\boldsymbol{\sigma}_{nM} + \delta\boldsymbol{\sigma}_{nM}^{\mathrm{T}}(\boldsymbol{\varepsilon}_{nG} - \boldsymbol{\varepsilon}_{nH})]\mathrm{d}v = \delta L_e \quad (24)$$

where $v$ denotes the multilayered body volume and $\delta L_e$ is the external virtual work. Subscript $H$ means that the quantities are calculated by using MFHL, subscript $G$ means that the strains are calculated by using the geometric relations (the relations which relate the strains to the derivative of the displacements) and subscript $M$ means that the transverse stresses are modeled with an axiomatic approach.

## 7. Governing equations

Consider layer $k$ of a multilayered plate composed of $N_l$ layers. The Reissner equation, in the case of pressures applied at the top and bottom of each layer $k$, is

$$\begin{aligned} &\int_{\Omega^k}\int_{z_{\mathrm{bot}_k}}^{z_{\mathrm{top}_k}} [\delta\boldsymbol{\varepsilon}_{pG}^{kT}\boldsymbol{\sigma}_{pH}^k + \delta\boldsymbol{\varepsilon}_{nG}^{kT}\boldsymbol{\sigma}_{nM}^k + \delta\boldsymbol{\sigma}_{nM}^{kT}(\boldsymbol{\varepsilon}_{nG}^k - \boldsymbol{\varepsilon}_{nH}^k)]\mathrm{d}z\mathrm{d}x\mathrm{d}y = \\ &\quad + \int_{\Omega^k} \delta u_x^k(x,y,z=z_{\mathrm{top}_k})P_x^{k\,\mathrm{t}}(x,y,z=z_{\mathrm{top}_k})\mathrm{d}x\mathrm{d}y \\ &\quad + \int_{\Omega^k} \delta u_y^k(x,y,z=z_{\mathrm{top}_k})P_y^{k\,\mathrm{t}}(x,y,z=z_{\mathrm{top}_k})\mathrm{d}x\mathrm{d}y \\ &\quad + \int_{\Omega^k} \delta u_z^k(x,y,z=z_{\mathrm{top}_k})P_z^{k\,\mathrm{t}}(x,y,z=z_{\mathrm{top}_k})\mathrm{d}x\mathrm{d}y \\ &\quad + \int_{\Omega^k} \delta u_x^k(x,y,z=z_{\mathrm{bot}_k})P_x^{k\,\mathrm{b}}(x,y,z=z_{\mathrm{bot}_k})\mathrm{d}x\mathrm{d}y \\ &\quad + \int_{\Omega^k} \delta u_y^k(x,y,z=z_{\mathrm{bot}_k})P_y^{k\,\mathrm{b}}(x,y,z=z_{\mathrm{bot}_k})\mathrm{d}x\mathrm{d}y \\ &\quad + \int_{\Omega^k} \delta u_z^k(x,y,z=z_{\mathrm{bot}_k})P_z^{k\,\mathrm{b}}(x,y,z=z_{\mathrm{bot}_k})\mathrm{d}x\mathrm{d}y \\ &\quad + \int_{\Gamma_\sigma^k}\int_{z_{\mathrm{bot}_k}}^{z_{\mathrm{top}_k}} [\delta u_n^k \overline{\sigma}_{nn}^{\,k} + \delta u_s^k \overline{\sigma}_{ns}^{\,k} + \delta u_z^k \overline{\sigma}_{nz}^{\,k}]\mathrm{d}z\mathrm{d}s \end{aligned} \quad (25)$$

where $P_x^{k\,\mathrm{t}}, P_y^{k\,\mathrm{t}}$ and $P_z^{k\,\mathrm{t}}$ are the top pressures (applied at $z = z_{\mathrm{top}_k}$), $P_x^{k\,\mathrm{b}}, P_y^{k\,\mathrm{b}}$ and $P_z^{k\,\mathrm{b}}$ are the bottom pressures (applied at $z = z_{\mathrm{bot}_k}$), $\overline{\sigma}_{nn}^{\,k}, \overline{\sigma}_{ns}^{\,k}$ and $\overline{\sigma}_{nz}^{\,k}$ are the specified normal and tangential components measured per unit area, $u_n^k, u_s^k$ and $u_z^k$ are the normal and tangential displacements on the edge $\Gamma_\sigma^k$ in which the stresses are specified. It is assumed that $\Omega^k = \Omega$. The geometrical relations can be rewritten using the formalism introduced for the generalized unified formulation as follows:

$$\begin{aligned} \varepsilon_{xx}^k &= \frac{\partial u_x^k}{\partial x} = {}^xF_{\alpha_{ux}}\, u_{x\alpha_{ux},x}^k \\ \varepsilon_{yy}^k &= \frac{\partial u_y^k}{\partial y} = {}^yF_{\alpha_{uy}}\, u_{y\alpha_{uy},y}^k \\ \gamma_{xy}^k &= \frac{\partial u_x^k}{\partial y} + \frac{\partial u_y^k}{\partial x} = {}^xF_{\alpha_{ux}}\, u_{x\alpha_{ux},y}^k + {}^yF_{\alpha_{uy}}\, u_{y\alpha_{uy},x}^k \\ \gamma_{xz}^k &= \frac{\partial u_z^k}{\partial x} + \frac{\partial u_x^k}{\partial z} = {}^zF_{\alpha_{uz}}\, u_{z\alpha_{uz},x}^k + {}^xF_{\alpha_{ux},z}\, u_{x\alpha_{ux}}^k \\ \gamma_{yz}^k &= \frac{\partial u_z^k}{\partial y} + \frac{\partial u_y^k}{\partial z} = {}^zF_{\alpha_{uz}}\, u_{z\alpha_{uz},y}^k + {}^yF_{\alpha_{uy},z}\, u_{y\alpha_{uy}}^k \\ \varepsilon_{zz}^k &= \frac{\partial u_z^k}{\partial z} = {}^zF_{\alpha_{uz},z}\, u_{z\alpha_{uz}}^k \end{aligned} \quad (26)$$

The out-of-plane stresses can be expanded along the thickness using GUF (see Eq. 10):

$$\begin{aligned} \sigma_{zx}^k &= s_x^k = {}^x\mathcal{F}_{\alpha_{sx}}\, s_{x\alpha_{sx}}^k \\ \sigma_{zy}^k &= s_y^k = {}^y\mathcal{F}_{\alpha_{sy}}\, s_{y\alpha_{sy}}^k \\ \sigma_{zz}^k &= s_z^k = {}^z\mathcal{F}_{\alpha_{sz}}\, s_{z\alpha_{sz}}^k \end{aligned} \quad (27)$$

To elaborate the terms of Eq. (25), mixed form of Hookes' law (Eq. 23) is used

$$\begin{aligned} \delta\boldsymbol{\varepsilon}_{pG}^{kT}\boldsymbol{\sigma}_{pH}^k = {} & \delta\varepsilon_{xx}^k C_{11}^k \varepsilon_{xx}^k + \delta\varepsilon_{xx}^k C_{12}^k \varepsilon_{yy}^k + \delta\varepsilon_{xx}^k C_{16}^k \gamma_{xy}^k + \delta\varepsilon_{xx}^k C_{13}^k \sigma_{zz}^k \\ & + \delta\varepsilon_{yy}^k C_{12}^k \varepsilon_{xx}^k + \delta\varepsilon_{yy}^k C_{22}^k \varepsilon_{yy}^k + \delta\varepsilon_{yy}^k C_{26}^k \gamma_{xy}^k + \delta\varepsilon_{yy}^k C_{23}^k \sigma_{zz}^k \\ & + \delta\gamma_{xy}^k C_{16}^k \varepsilon_{xx}^k + \delta\gamma_{xy}^k C_{26}^k \varepsilon_{yy}^k + \delta\gamma_{xy}^k C_{66}^k \gamma_{xy}^k + \delta\gamma_{xy}^k C_{36}^k \sigma_{zz}^k \end{aligned} \quad (28)$$

$$\boldsymbol{\varepsilon}_{nG}^{kT}\boldsymbol{\sigma}_{nM}^k = \delta\gamma_{xz}^k s_x^k + \delta\gamma_{yz}^k s_y^k + \delta\varepsilon_{zz}^k s_z^k \quad (29)$$

$$\boldsymbol{\sigma}_{nM}^{kT}\boldsymbol{\varepsilon}_{nG}^k = \delta s_x^k \gamma_{xz}^k + \delta s_y^k \gamma_{yz}^k + \delta s_z^k \varepsilon_{zz}^k \quad (30)$$

$$\begin{aligned} -\delta\boldsymbol{\sigma}_{nM}^{kT}\boldsymbol{\varepsilon}_{nH}^k = {} & -\delta s_x^k C_{55}^k s_x^k - \delta s_x^k C_{45}^k s_y^k - \delta s_y^k C_{45}^k s_x^k - \delta s_y^k C_{44}^k s_y^k \\ & + \delta s_z^k C_{13}^k \varepsilon_{xx}^k + \delta s_z^k C_{23}^k \varepsilon_{yy}^k + \delta s_z^k C_{36}^k \gamma_{xy}^k - \delta s_z^k C_{33}^k s_z^k \end{aligned} \quad (31)$$

Using the geometric relations (Eq. 26) and the expressions adopted for the stresses (Eq. 27):

$$\begin{aligned} \delta\boldsymbol{\varepsilon}_{pG}^{kT}\boldsymbol{\sigma}_{pH}^k = {} & C_{11}^k\, {}^xF_{\alpha_{ux}}\, {}^xF_{\beta_{ux}}\, \delta u_{x\alpha_{ux},x}^k\, u_{x\beta_{ux},x}^k + C_{12}^k\, {}^xF_{\alpha_{ux}}\, {}^yF_{\beta_{uy}}\, \delta u_{x\alpha_{ux},x}^k\, u_{y\beta_{uy},y}^k \\ & + C_{16}^k\, {}^xF_{\alpha_{ux}}\, {}^xF_{\beta_{ux}}\, \delta u_{x\alpha_{ux},x}^k\, u_{x\beta_{ux},y}^k + C_{16}^k\, {}^xF_{\alpha_{ux}}\, {}^yF_{\beta_{uy}}\, \delta u_{x\alpha_{ux},x}^k\, u_{y\beta_{uy},x}^k \\ & + C_{13}^k\, {}^xF_{\alpha_{ux}}\, {}^z\mathcal{F}_{\beta_{sz}}\, \delta u_{x\alpha_{ux},x}^k\, s_{z\beta_{sz}}^k + C_{12}^k\, {}^yF_{\alpha_{uy}}\, {}^xF_{\beta_{ux}}\, \delta u_{y\alpha_{uy},y}^k\, u_{x\beta_{ux},x}^k \\ & + C_{22}^k\, {}^yF_{\alpha_{uy}}\, {}^yF_{\beta_{uy}}\, \delta u_{y\alpha_{uy},y}^k\, u_{y\beta_{uy},y}^k + C_{26}^k\, {}^yF_{\alpha_{uy}}\, {}^xF_{\beta_{ux}}\, \delta u_{y\alpha_{uy},y}^k\, u_{x\beta_{ux},y}^k \\ & + C_{26}^k\, {}^yF_{\alpha_{uy}}\, {}^yF_{\beta_{uy}}\, \delta u_{y\alpha_{uy},y}^k\, u_{y\beta_{uy},x}^k + C_{23}^k\, {}^yF_{\alpha_{uy}}\, {}^z\mathcal{F}_{\beta_{sz}}\, \delta u_{y\alpha_{uy},y}^k\, s_{z\beta_{sz}}^k \\ & + C_{16}^k\, {}^xF_{\alpha_{ux}}\, {}^xF_{\beta_{ux}}\, \delta u_{x\alpha_{ux},y}^k\, u_{x\beta_{ux},x}^k + C_{16}^k\, {}^yF_{\alpha_{uy}}\, {}^xF_{\beta_{ux}}\, \delta u_{y\alpha_{uy},x}^k\, u_{x\beta_{ux},x}^k \\ & + C_{26}^k\, {}^xF_{\alpha_{ux}}\, {}^yF_{\beta_{uy}}\, \delta u_{x\alpha_{ux},y}^k\, u_{y\beta_{uy},y}^k + C_{26}^k\, {}^yF_{\alpha_{uy}}\, {}^yF_{\beta_{uy}}\, \delta u_{y\alpha_{uy},x}^k\, u_{y\beta_{uy},y}^k \\ & + C_{66}^k\, {}^xF_{\alpha_{ux}}\, {}^xF_{\beta_{ux}}\, \delta u_{x\alpha_{ux},y}^k\, u_{x\beta_{ux},y}^k + C_{66}^k\, {}^yF_{\alpha_{uy}}\, {}^xF_{\beta_{ux}}\, \delta u_{y\alpha_{uy},x}^k\, u_{x\beta_{ux},y}^k \\ & + C_{66}^k\, {}^xF_{\alpha_{ux}}\, {}^yF_{\beta_{uy}}\, \delta u_{x\alpha_{ux},y}^k\, u_{y\beta_{uy},x}^k + C_{66}^k\, {}^yF_{\alpha_{uy}}\, {}^yF_{\beta_{uy}}\, \delta u_{y\alpha_{uy},x}^k\, u_{y\beta_{uy},x}^k \\ & + C_{36}^k\, {}^xF_{\alpha_{ux}}\, {}^z\mathcal{F}_{\beta_{sz}}\, \delta u_{x\alpha_{ux},y}^k\, s_{z\beta_{sz}}^k + C_{36}^k\, {}^yF_{\alpha_{uy}}\, {}^z\mathcal{F}_{\beta_{sz}}\, \delta u_{y\alpha_{uy},x}^k\, s_{z\beta_{sz}}^k \end{aligned} \quad (32)$$

$$\begin{aligned} \delta\boldsymbol{\varepsilon}_{nG}^{kT}\boldsymbol{\sigma}_{nM}^k = {} & {}^zF_{\alpha_{uz}}\, {}^x\mathcal{F}_{\beta_{sx}}\, \delta u_{z\alpha_{uz},x}^k\, s_{x\beta_{sx}}^k + {}^xF_{\alpha_{ux},z}\, {}^x\mathcal{F}_{\beta_{sx}}\, \delta u_{x\alpha_{ux}}^k\, s_{x\beta_{sx}}^k \\ & + {}^zF_{\alpha_{uz}}\, {}^y\mathcal{F}_{\beta_{sy}}\, \delta u_{z\alpha_{uz},y}^k\, s_{y\beta_{sy}}^k + {}^yF_{\alpha_{uy},z}\, {}^y\mathcal{F}_{\beta_{sy}}\, \delta u_{y\alpha_{uy}}^k\, s_{y\beta_{sy}}^k \\ & + {}^zF_{\alpha_{uz},z}\, {}^z\mathcal{F}_{\beta_{sz}}\, \delta u_{z\alpha_{uz}}^k\, s_{z\beta_{sz}}^k \end{aligned} \quad (33)$$

$$\begin{aligned}\delta\boldsymbol{\sigma}^{kT}_{nM}\boldsymbol{\varepsilon}^{k}_{nG} &= {}^{x}\mathcal{F}_{\alpha_{sx}}\,{}^{z}F_{\beta_{uz}}\,\delta s^{k}_{x\alpha_{sx}}\,u^{k}_{z\beta_{uz},x} + {}^{x}\mathcal{F}_{\alpha_{sx}}\,{}^{x}F_{\beta_{ux},z}\,\delta s^{k}_{x\alpha_{sx}}\,u^{k}_{x\beta_{ux}}\\ &+ {}^{y}\mathcal{F}_{\alpha_{sy}}\,{}^{z}F_{\beta_{uz}}\,\delta s^{k}_{y\alpha_{sy}}\,u^{k}_{z\beta_{uz},y} + {}^{y}\mathcal{F}_{\alpha_{sy}}\,{}^{y}F_{\beta_{uy},z}\,\delta s^{k}_{y\alpha_{sy}}\,u^{k}_{y\beta_{uy}}\\ &+ {}^{z}\mathcal{F}_{\alpha_{sz}}\,{}^{z}F_{\beta_{uz},z}\,\delta s^{k}_{z\alpha_{sz}}\,u^{k}_{z\beta_{uz}} \end{aligned} \tag{34}$$

$$\begin{aligned}-\delta\boldsymbol{\sigma}^{kT}_{nM}\boldsymbol{\varepsilon}^{k}_{nH} &= -C^{k}_{55}\,{}^{x}\mathcal{F}_{\alpha_{sx}}\,{}^{x}\mathcal{F}_{\beta_{sx}}\,\delta s^{k}_{x\alpha_{sx}}\,s^{k}_{x\beta_{sx}} - C^{k}_{45}\,{}^{x}\mathcal{F}_{\alpha_{sx}}\,{}^{y}\mathcal{F}_{\beta_{sy}}\,\delta s^{k}_{x\alpha_{sx}}\,s^{k}_{y\beta_{sy}}\\ &- C^{k}_{45}\,{}^{y}\mathcal{F}_{\alpha_{sy}}\,{}^{x}\mathcal{F}_{\beta_{sx}}\,\delta s^{k}_{y\alpha_{sy}}\,s^{k}_{x\beta_{sx}} - C^{k}_{44}\,{}^{y}\mathcal{F}_{\alpha_{sy}}\,{}^{y}\mathcal{F}_{\beta_{sy}}\,\delta s^{k}_{y\alpha_{sy}}\,s^{k}_{y\beta_{sy}}\\ &+ C^{k}_{13}\,{}^{z}\mathcal{F}_{\alpha_{sz}}\,{}^{x}F_{\beta_{ux}}\,\delta s^{k}_{z\alpha_{sz}}\,u^{k}_{x\beta_{ux},x} + C^{k}_{23}\,{}^{z}\mathcal{F}_{\alpha_{sz}}\,{}^{y}F_{\beta_{uy}}\,\delta s^{k}_{z\alpha_{sz}}\,u^{k}_{y\beta_{uy},y}\\ &+ C^{k}_{36}\,{}^{z}\mathcal{F}_{\alpha_{sz}}\,{}^{x}F_{\beta_{ux}}\,\delta s^{k}_{z\alpha_{sz}}\,u^{k}_{x\beta_{ux},y} + C^{k}_{36}\,{}^{z}\mathcal{F}_{\alpha_{sz}}\,{}^{y}F_{\beta_{uy}}\,\delta s^{k}_{z\alpha_{sz}}\,u^{k}_{y\beta_{uy},x}\\ &- C^{k}_{33}\,{}^{z}\mathcal{F}_{\alpha_{sz}}\,{}^{z}\mathcal{F}_{\beta_{sz}}\,\delta s^{k}_{z\alpha_{sz}}\,s^{k}_{z\beta_{sz}} \end{aligned} \tag{35}$$

Therefore, it can be inferred (see Fig. 4 for the definitions related to the integrals along the thickness) that

$$\begin{aligned}&\int_{\Omega^k}\int_{z_{\mathrm{bot}_k}}^{z_{\mathrm{top}_k}} \delta\boldsymbol{\varepsilon}^{kT}_{pG}\boldsymbol{\sigma}^{k}_{pH}\,\mathrm{d}z\mathrm{d}x\mathrm{d}y\\ &= +Z^{k\,\alpha_{ux}\beta_{ux}}_{11\,u_x u_x}\int_{\Omega^k}\delta u^{k}_{x\alpha_{ux},x}u^{k}_{x\beta_{ux},x}\,\mathrm{d}x\mathrm{d}y + Z^{k\,\alpha_{ux}\beta_{uy}}_{12\,u_x u_y}\int_{\Omega^k}\delta u^{k}_{x\alpha_{ux},x}u^{k}_{y\beta_{uy},y}\,\mathrm{d}x\mathrm{d}y\\ &+ Z^{k\,\alpha_{ux}\beta_{ux}}_{16\,u_x u_x}\int_{\Omega^k}\delta u^{k}_{x\alpha_{ux},x}u^{k}_{x\beta_{ux},y}\,\mathrm{d}x\mathrm{d}y + Z^{k\,\alpha_{ux}\beta_{uy}}_{16\,u_x u_y}\int_{\Omega^k}\delta u^{k}_{x\alpha_{ux},x}u^{k}_{y\beta_{uy},x}\,\mathrm{d}x\mathrm{d}y\\ &+ Z^{k\,\alpha_{ux}\beta_{sz}}_{13\,u_x s_z}\int_{\Omega^k}\delta u^{k}_{x\alpha_{ux},x}s^{k}_{z\beta_{sz}}\,\mathrm{d}x\mathrm{d}y + Z^{k\,\alpha_{uy}\beta_{ux}}_{12\,u_y u_x}\int_{\Omega^k}\delta u^{k}_{y\alpha_{uy},y}u^{k}_{x\beta_{ux},x}\,\mathrm{d}x\mathrm{d}y\\ &+ Z^{k\,\alpha_{uy}\beta_{uy}}_{22\,u_y u_y}\int_{\Omega^k}\delta u^{k}_{y\alpha_{uy},y}u^{k}_{y\beta_{uy},y}\,\mathrm{d}x\mathrm{d}y + Z^{k\,\alpha_{uy}\beta_{ux}}_{26\,u_y u_x}\int_{\Omega^k}\delta u^{k}_{y\alpha_{uy},y}u^{k}_{x\beta_{ux},y}\,\mathrm{d}x\mathrm{d}y\\ &+ Z^{k\,\alpha_{uy}\beta_{uy}}_{26\,u_y u_y}\int_{\Omega^k}\delta u^{k}_{y\alpha_{uy},y}u^{k}_{y\beta_{uy},x}\,\mathrm{d}x\mathrm{d}y + Z^{k\,\alpha_{uy}\beta_{sz}}_{23\,u_y s_z}\int_{\Omega^k}\delta u^{k}_{y\alpha_{uy},y}s^{k}_{z\beta_{sz}}\,\mathrm{d}x\mathrm{d}y\\ &+ Z^{k\,\alpha_{ux}\beta_{ux}}_{16\,u_x u_x}\int_{\Omega^k}\delta u^{k}_{x\alpha_{ux},y}u^{k}_{x\beta_{ux},x}\,\mathrm{d}x\mathrm{d}y + Z^{k\,\alpha_{uy}\beta_{ux}}_{16\,u_y u_x}\int_{\Omega^k}\delta u^{k}_{y\alpha_{uy},x}u^{k}_{x\beta_{ux},x}\,\mathrm{d}x\mathrm{d}y\\ &+ Z^{k\,\alpha_{ux}\beta_{uy}}_{26\,u_x u_y}\int_{\Omega^k}\delta u^{k}_{x\alpha_{ux},y}u^{k}_{y\beta_{uy},y}\,\mathrm{d}x\mathrm{d}y + Z^{k\,\alpha_{uy}\beta_{uy}}_{26\,u_y u_y}\int_{\Omega^k}\delta u^{k}_{y\alpha_{uy},x}u^{k}_{y\beta_{uy},y}\,\mathrm{d}x\mathrm{d}y\\ &+ Z^{k\,\alpha_{ux}\beta_{ux}}_{66\,u_x u_x}\int_{\Omega^k}\delta u^{k}_{x\alpha_{ux},y}u^{k}_{x\beta_{ux},y}\,\mathrm{d}x\mathrm{d}y + Z^{k\,\alpha_{uy}\beta_{ux}}_{66\,u_y u_x}\int_{\Omega^k}\delta u^{k}_{y\alpha_{uy},x}u^{k}_{x\beta_{ux},y}\,\mathrm{d}x\mathrm{d}y\\ &+ Z^{k\,\alpha_{ux}\beta_{uy}}_{66\,u_x u_y}\int_{\Omega^k}\delta u^{k}_{x\alpha_{ux},y}u^{k}_{y\beta_{uy},x}\,\mathrm{d}x\mathrm{d}y + Z^{k\,\alpha_{uy}\beta_{uy}}_{66\,u_y u_y}\int_{\Omega^k}\delta u^{k}_{y\alpha_{uy},x}u^{k}_{y\beta_{uy},x}\,\mathrm{d}x\mathrm{d}y\\ &+ Z^{k\,\alpha_{ux}\beta_{sz}}_{36\,u_x s_z}\int_{\Omega^k}\delta u^{k}_{x\alpha_{ux},y}s^{k}_{z\beta_{sz}}\,\mathrm{d}x\mathrm{d}y + Z^{k\,\alpha_{uy}\beta_{sz}}_{36\,u_y s_z}\int_{\Omega^k}\delta u^{k}_{y\alpha_{uy},x}s^{k}_{z\beta_{sz}}\,\mathrm{d}x\mathrm{d}y \end{aligned} \tag{36}$$

$$\begin{aligned}&\int_{\Omega^k}\int_{z_{\mathrm{bot}_k}}^{z_{\mathrm{top}_k}} \delta\boldsymbol{\varepsilon}^{kT}_{nG}\boldsymbol{\sigma}^{k}_{nM}\,\mathrm{d}z\mathrm{d}x\mathrm{d}y\\ &= +Z^{k\,\alpha_{uz}\beta_{sx}}_{u_z s_x}\int_{\Omega^k}\delta u^{k}_{z\alpha_{uz},x}s^{k}_{x\beta_{sx}}\,\mathrm{d}x\mathrm{d}y + Z^{k\,\alpha_{ux},z\beta_{sx}}_{u_x s_x}\int_{\Omega^k}\delta u^{k}_{x\alpha_{ux}}s^{k}_{x\beta_{sx}}\,\mathrm{d}x\mathrm{d}y\\ &+ Z^{k\,\alpha_{uz}\beta_{sy}}_{u_z s_y}\int_{\Omega^k}\delta u^{k}_{z\alpha_{uz},y}s^{k}_{y\beta_{sy}}\,\mathrm{d}x\mathrm{d}y + Z^{k\,\alpha_{uy},z\beta_{sy}}_{u_y s_y}\int_{\Omega^k}\delta u^{k}_{y\alpha_{uy}}s^{k}_{y\beta_{sy}}\,\mathrm{d}x\mathrm{d}y\\ &+ Z^{k\,\alpha_{uz},z\beta_{sz}}_{u_z s_z}\int_{\Omega^k}\delta u^{k}_{z\alpha_{uz}}s^{k}_{z\beta_{sz}}\,\mathrm{d}x\mathrm{d}y \end{aligned} \tag{37}$$

$$\begin{aligned}&\int_{\Omega^k}\int_{z_{\mathrm{bot}_k}}^{z_{\mathrm{top}_k}} \delta\boldsymbol{\sigma}^{kT}_{nM}\boldsymbol{\varepsilon}^{k}_{nG}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y = +Z^{k\,\alpha_{sx}\beta_{uz}}_{s_x u_z}\int_{\Omega^k}\delta s^{k}_{x\alpha_{sx}}u^{k}_{z\beta_{uz},x}\,\mathrm{d}x\mathrm{d}y\\ &+ Z^{k\,\alpha_{sx}\beta_{ux},z}_{s_x u_x}\int_{\Omega^k}\delta s^{k}_{x\alpha_{sx}}u^{k}_{x\beta_{ux}}\,\mathrm{d}x\mathrm{d}y + Z^{k\,\alpha_{sy}\beta_{uz}}_{s_y u_z}\int_{\Omega^k}\delta s^{k}_{y\alpha_{sy}}u^{k}_{z\beta_{uz},y}\,\mathrm{d}x\mathrm{d}y\\ &+ Z^{k\,\alpha_{sy}\beta_{uy},z}_{s_y u_y}\int_{\Omega^k}\delta s^{k}_{y\alpha_{sy}}u^{k}_{y\beta_{uy}}\,\mathrm{d}x\mathrm{d}y + Z^{k\,\alpha_{sz}\beta_{uz},z}_{s_z u_z}\int_{\Omega^k}\delta s^{k}_{z\alpha_{sz}}u^{k}_{z\beta_{uz}}\,\mathrm{d}x\mathrm{d}y \end{aligned} \tag{38}$$

$$\begin{aligned}&-\int_{\Omega^k}\int_{z_{\mathrm{bot}_k}}^{z_{\mathrm{top}_k}} \delta\boldsymbol{\sigma}^{kT}_{nM}\boldsymbol{\varepsilon}^{k}_{nH}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y\\ &= -Z^{k\,\alpha_{sx}\beta_{sx}}_{55\,s_x s_x}\int_{\Omega^k}\delta s^{k}_{x\alpha_{sx}}s^{k}_{x\beta_{sx}}\,\mathrm{d}x\mathrm{d}y - Z^{k\,\alpha_{sx}\beta_{sy}}_{45\,s_x s_y}\int_{\Omega^k}\delta s^{k}_{x\alpha_{sx}}s^{k}_{y\beta_{sy}}\,\mathrm{d}x\mathrm{d}y\\ &- Z^{k\,\alpha_{sy}\beta_{sx}}_{45\,s_y s_x}\int_{\Omega^k}\delta s^{k}_{y\alpha_{sy}}s^{k}_{x\beta_{sx}}\,\mathrm{d}x\mathrm{d}y - Z^{k\,\alpha_{sy}\beta_{sy}}_{44\,s_y s_y}\int_{\Omega^k}\delta s^{k}_{y\alpha_{sy}}s^{k}_{y\beta_{sy}}\,\mathrm{d}x\mathrm{d}y\\ &+ Z^{k\,\alpha_{sz}\beta_{ux}}_{13\,s_z u_x}\int_{\Omega^k}\delta s^{k}_{z\alpha_{sz}}u^{k}_{x\beta_{ux},x}\,\mathrm{d}x\mathrm{d}y + Z^{k\,\alpha_{sz}\beta_{uy}}_{23\,s_z u_y}\int_{\Omega^k}\delta s^{k}_{z\alpha_{sz}}u^{k}_{y\beta_{uy},y}\,\mathrm{d}x\mathrm{d}y\\ &+ Z^{k\,\alpha_{sz}\beta_{ux}}_{36\,s_z u_x}\int_{\Omega^k}\delta s^{k}_{z\alpha_{sz}}u^{k}_{x\beta_{ux},y}\,\mathrm{d}x\mathrm{d}y + Z^{k\,\alpha_{sz}\beta_{uy}}_{36\,s_z u_y}\int_{\Omega^k}\delta s^{k}_{z\alpha_{sz}}u^{k}_{y\beta_{uy},x}\,\mathrm{d}x\mathrm{d}y\\ &- Z^{k\,\alpha_{sz}\beta_{sz}}_{33\,s_z s_z}\int_{\Omega^k}\delta s^{k}_{z\alpha_{sz}}s^{k}_{z\beta_{sz}}\,\mathrm{d}x\mathrm{d}y \end{aligned} \tag{39}$$

The equations are elaborated. In particular, the terms $\int_{\Omega^k}\int_{z_{\mathrm{bot}_k}}^{z_{\mathrm{top}_k}}\delta\boldsymbol{\varepsilon}^{kT}_{pG}\boldsymbol{\sigma}^{k}_{pH}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$ and $\int_{\Omega^k}\int_{z_{\mathrm{bot}_k}}^{z_{\mathrm{top}_k}}\delta\boldsymbol{\varepsilon}^{kT}_{nG}\boldsymbol{\sigma}^{k}_{nM}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$ are integrated by parts [41]. Notice that the term $\int_{\Omega^k}\int_{z_{\mathrm{bot}_k}}^{z_{\mathrm{top}_k}}\delta\boldsymbol{\sigma}^{kT}_{nM}\boldsymbol{\varepsilon}^{k}_{nG}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$ does not contain any integral that needs to be transformed using integration by parts. Therefore, no boundary terms are generated by this term. The same consideration is valid for the term $-\int_{\Omega^k}\int_{z_{\mathrm{bot}_k}}^{z_{\mathrm{top}_k}}\delta\boldsymbol{\sigma}^{kT}_{nM}\boldsymbol{\varepsilon}^{k}_{nH}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$. The details of this integration by parts are omitted for brevity.

Now the focus is on the external virtual work. Consider the applied pressures. They are written in a slightly different manner in order to use the power of the present generalized unified

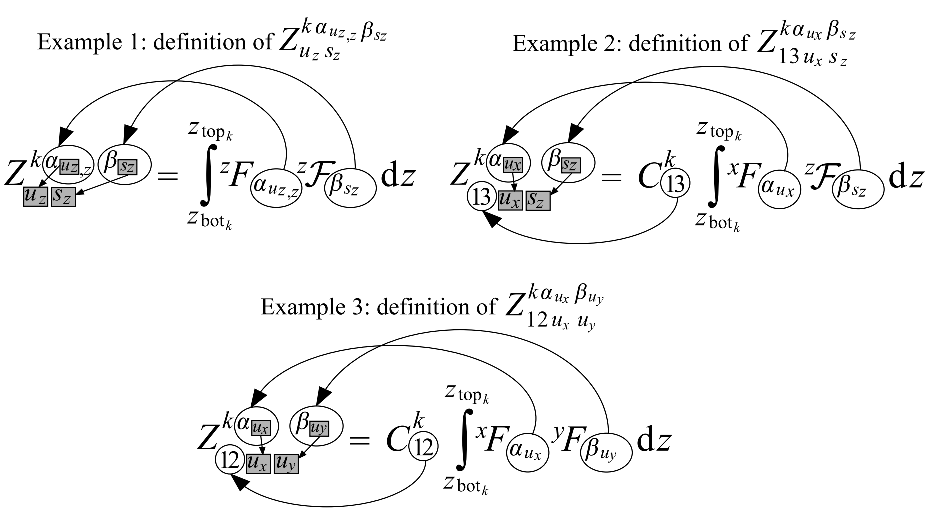


**Fig. 4.** GUF. Example of some definitions related to the integrals along the thickness.

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formulation. Consider in particular the pressure term $\delta u_x^k(x,y,z_{\text{top}_k}) P_x^{k\,t}(x,y,z_{\text{top}_k})$ (see Eq. 25). $P_x^{k\,t}(x,y,z_{\text{top}_k})$ is the force per unit of area in direction $x$ (subscript $x$) and applied at the *top* surface ($z = z_{\text{top}_k}$) of the layer $k$ (the superscript $t$ means "top"). The pressure $P_x^{k\,t}(x,y,z_{\text{top}_k})$ is thought as a function along the thickness (this is in order to use GUF) as follows:

$$P_x^{k\,t}(x,y,z_{\text{top}_k}) = {}^{x}F_{\alpha_{u_x}}(z = z_{\text{top}_k}) \cdot P_{x\alpha_{u_x}}^{k\,t}(x,y) \tag{40}$$

Notice that the same number of terms used for the expansion in the thickness direction of the displacement $u_x^k$ has been considered. This is a natural choice, considering the fact that the pressure in $x$ direction works with the displacement in $x$ direction.

Now the following notation is introduced:

$${}^{x}F^{\text{t}}_{\alpha_{u_x}} \doteq {}^{x}F_{\alpha_{u_x}}(z = z_{\text{top}_k}) \tag{41}$$

The superscript $t$ means that the functions ${}^{x}F_{\alpha_{u_x}}$ are *all* calculated at the $z$ which corresponds to the top surface of layer $k$ (i.e., $z = z_{\text{top}_k}$) *where the pressure is supposed to be applied*. Using this definition, the pressure can be written using the formalism used for the displacements:

$$P_x^{k\,t}(x,y,z_{\text{top}_k}) = {}^{x}F^{\text{t}}_{\alpha_{u_x}} \cdot P_{x\alpha_{u_x}}^{k\,t}(x,y) = {}^{x}F^{\text{t}}_{\alpha_{u_x}} P_{x\alpha_{u_x}}^{k\,t} \tag{42}$$

Observing that

$$\delta u_x^k(x,y,z_{\text{top}_k}) = {}^{x}F_{\alpha_{u_x}}(z = z_{\text{top}_k}) \cdot \delta u_{x\alpha_{u_x}}^k(x,y) = {}^{x}F^{\text{t}}_{\alpha_{u_x}} \delta u_{x\alpha_{u_x}}^k \tag{43}$$

the contribution of the pressure $P_x^{k\,t}(x,y,z_{\text{top}_k})$ to the external virtual work can be written as (the secondary master index $\beta$ needs to be used):

$$\delta u_x^k(x,y,z_{\text{top}_k}) P_x^{k\,t}(x,y,z_{\text{top}_k}) = \delta u_{x\alpha_{u_x}}^k \, {}^{x}F^{\text{t}}_{\alpha_{u_x}} \, {}^{x}F^{\text{t}}_{\beta_{u_x}} P_{x\beta_{u_x}}^{k\,t} \tag{44}$$

or

$$\delta u_x^k(x,y,z_{\text{top}_k}) P_x^{k\,t}(x,y,z_{\text{top}_k}) = \delta u_{x\alpha_{u_x}}^k \, {}^{t}D_{u_x u_x}^{k\,\alpha_{u_x}\beta_{u_x}} P_{x\beta_{u_x}}^{k\,t} \tag{45}$$

where

$${}^{t}D_{u_x u_x}^{k\,\alpha_{u_x}\beta_{u_x}} = {}^{x}F^{\text{t}}_{\alpha_{u_x}} \, {}^{x}F^{\text{t}}_{\beta_{u_x}} \tag{46}$$

The other applied pressures are similarly treated. Superscript b is used to indicate the bottom pressure; for example, ${}^{x}F^{b}_{\alpha_{u_x}} \doteq {}^{x}F_{\alpha_{u_x}}(z = z_{\text{bot}_k})$). The virtual external work can then be rewritten as

$$\begin{aligned}
\delta L_e^k = &+ \int_{\Omega^k} \delta u_{x\alpha_{u_x}} \, {}^{t}D_{u_x u_x}^{k\,\alpha_{u_x}\beta_{u_x}} P_{x\beta_{u_x}}^{k\,t} \, \mathrm{d}x\mathrm{d}y + \int_{\Omega^k} \delta u_{y\alpha_{u_y}} \, {}^{t}D_{u_y u_y}^{k\,\alpha_{u_y}\beta_{u_y}} P_{y\beta_{u_y}}^{k\,t} \, \mathrm{d}x\mathrm{d}y \\
&+ \int_{\Omega^k} \delta u_{z\alpha_{u_z}} \, {}^{t}D_{u_z u_z}^{k\,\alpha_{u_z}\beta_{u_z}} P_{z\beta_{u_z}}^{k\,t} \, \mathrm{d}x\mathrm{d}y + \int_{\Omega^k} \delta u_{x\alpha_{u_x}} \, {}^{b}D_{u_x u_x}^{k\,\alpha_{u_x}\beta_{u_x}} P_{x\beta_{u_x}}^{k\,b} \, \mathrm{d}x\mathrm{d}y \\
&+ \int_{\Omega^k} \delta u_{y\alpha_{u_y}} \, {}^{b}D_{u_y u_y}^{k\,\alpha_{u_y}\beta_{u_y}} P_{y\beta_{u_y}}^{k\,b} \, \mathrm{d}x\mathrm{d}y + \int_{\Omega^k} \delta u_{z\alpha_{u_z}} \, {}^{b}D_{u_z u_z}^{k\,\alpha_{u_z}\beta_{u_z}} P_{z\beta_{u_z}}^{k\,b} \, \mathrm{d}x\mathrm{d}y \\
&+ \int_{\Gamma_\sigma^k} \int_{z_{\text{bot}_k}}^{z_{\text{top}_k}} [\delta u_n^k \overline{\sigma}_{nn}^k + \delta u_s^k \overline{\sigma}_{ns}^k + \delta u_z^k \overline{\sigma}_{nz}^k] \mathrm{d}z \, \mathrm{d}s
\end{aligned} \tag{47}$$

The pressure terms (for example $P_{x\beta_{u_x}}^{k\,t}$) are inputs of the problem. Terms of the type ${}^{t}D_{u_x u_x}^{k\,\alpha_{u_x}\beta_{u_x}}$ are defined as *pressure fundamental kernels* and are $1 \times 1$ matrices.

The pressure has been treated in a particular form because it is a natural choice for the case of multilayered structures analyzed with GUF.

The term in Eq. (47), which contains the prescribed stresses, is elaborated to obtain an expression that contains the displacements $u_x^k, u_y^k$ and $u_z^k$. This elaboration is required since, in general, different orders of expansions are used for different variables. After a few elaborations it can be demonstrated that the contribution of layer $k$ to the expression of virtual external work is

$$\begin{aligned}
\delta L_e^k = &+ \int_{\Omega^k} \delta u_{x\alpha_{u_x}}^k \, {}^{t}D_{u_x u_x}^{k\,\alpha_{u_x}\beta_{u_x}} P_{x\beta_{u_x}}^{k\,t} \, \mathrm{d}x\mathrm{d}y + \int_{\Omega^k} \delta u_{y\alpha_{u_y}}^k \, {}^{t}D_{u_y u_y}^{k\,\alpha_{u_y}\beta_{u_y}} P_{y\beta_{u_y}}^{k\,t} \, \mathrm{d}x\mathrm{d}y \\
&+ \int_{\Omega^k} \delta u_{z\alpha_{u_z}}^k \, {}^{t}D_{u_z u_z}^{k\,\alpha_{u_z}\beta_{u_z}} P_{z\beta_{u_z}}^{k\,t} \, \mathrm{d}x\mathrm{d}y + \int_{\Omega^k} \delta u_{x\alpha_{u_x}}^k \, {}^{b}D_{u_x u_x}^{k\,\alpha_{u_x}\beta_{u_x}} P_{x\beta_{u_x}}^{k\,b} \, \mathrm{d}x\mathrm{d}y \\
&+ \int_{\Omega^k} \delta u_{y\alpha_{u_y}}^k \, {}^{b}D_{u_y u_y}^{k\,\alpha_{u_y}\beta_{u_y}} P_{y\beta_{u_y}}^{k\,b} \, \mathrm{d}x\mathrm{d}y + \int_{\Omega^k} \delta u_{z\alpha_{u_z}}^k \, {}^{b}D_{u_z u_z}^{k\,\alpha_{u_z}\beta_{u_z}} P_{z\beta_{u_z}}^{k\,b} \, \mathrm{d}x\mathrm{d}y \\
&+ Z_{u_x u_x}^{k\,\alpha_{u_x}\beta_{u_x}} \int_{\Gamma_\sigma^{k\,x}} \delta u_{x\alpha_{u_x}}^k \, \bar{t}_{x\beta_{u_x}}^k \, \mathrm{d}s + Z_{u_y u_y}^{k\,\alpha_{u_y}\beta_{u_y}} \int_{\Gamma_\sigma^{k\,y}} \delta u_{y\alpha_{u_y}}^k \, \bar{t}_{y\beta_{u_y}}^k \, \mathrm{d}s \\
&+ Z_{u_z u_z}^{k\,\alpha_{u_z}\beta_{u_z}} \int_{\Gamma_\sigma^{k\,z}} \delta u_{z\alpha_{u_z}}^k \, \bar{t}_{z\beta_{u_z}}^k \, \mathrm{d}s
\end{aligned} \tag{48}$$

It is now assumed that where $\bar{t}_{x\alpha_{u_x}}^k$ *is assigned* the corresponding quantity $u_{x\alpha_{u_x}}^k$ *is not assigned.* Similarly, where $\bar{t}_{y\alpha_{u_y}}^k$ *is assigned* $u_{y\alpha_{u_y}}^k$ *is not assigned* and where $\bar{t}_{z\alpha_{u_z}}^k$ *is assigned* $u_{z\alpha_{u_z}}^k$ *is not assigned*. When the displacements are assigned, the virtual variations of the displacements are zero. For example, consider the displacement $u_{x\alpha_{u_x}}^k$, which is assigned (see above) only on the boundary portion $\Gamma^k - \Gamma_\sigma^{k\,x}$. Therefore, in this portion of the boundary $\delta u_{x\alpha_{u_x}}^k = 0$. Similar considerations can be made for the other displacements and, thus, the following relations can be written:

$$\begin{aligned}
\delta u_{x\alpha_{u_x}}^k &= 0 \quad \text{on} \quad \Gamma^k - \Gamma_\sigma^{k\,x} \\
\delta u_{y\alpha_{u_y}}^k &= 0 \quad \text{on} \quad \Gamma^k - \Gamma_\sigma^{k\,y} \\
\delta u_{z\alpha_{u_z}}^k &= 0 \quad \text{on} \quad \Gamma^k - \Gamma_\sigma^{k\,z}
\end{aligned} \tag{49}$$

Considering these last relations, the governing equations are:

$$\begin{aligned}
\delta u_{x\alpha_{u_x}}^k :& -Z_{11\,u_x u_x}^{k\,\alpha_{u_x}\beta_{u_x}} u_{x\beta_{u_x},x,x}^k - Z_{12\,u_x u_y}^{k\,\alpha_{u_x}\beta_{u_y}} u_{y\beta_{u_y},y,x}^k - Z_{16\,u_x u_x}^{k\,\alpha_{u_x}\beta_{u_x}} u_{x\beta_{u_x},y,x}^k \\
& - Z_{16\,u_x u_y}^{k\,\alpha_{u_x}\beta_{u_y}} u_{y\beta_{u_y},x,x}^k - Z_{13\,u_x s_z}^{k\,\alpha_{u_x}\beta_{s_z}} s_{z\beta_{s_z},x}^k - Z_{16\,u_x u_x}^{k\,\alpha_{u_x}\beta_{u_x}} u_{x\beta_{u_x},x,y}^k - Z_{26\,u_x u_y}^{k\,\alpha_{u_x}\beta_{u_y}} u_{y\beta_{u_y},y,y}^k \\
& - Z_{66\,u_x u_x}^{k\,\alpha_{u_x}\beta_{u_x}} u_{x\beta_{u_x},y,y}^k - Z_{66\,u_x u_y}^{k\,\alpha_{u_x}\beta_{u_y}} u_{y\beta_{u_y},x,y}^k - Z_{36\,u_x s_z}^{k\,\alpha_{u_x}\beta_{s_z}} s_{z\beta_{s_z},y}^k \\
& + Z_{u_x s_x}^{k\,\alpha_{u_x},z\beta_{s_x}} s_{x\beta_{s_x}}^k - D_{u_x u_x}^{k\,t\,\alpha_{u_x}\beta_{u_x}} P_{x\beta_{u_x}}^{k\,\text{t}} - D_{u_x u_x}^{k\,b\,\alpha_{u_x}\beta_{u_x}} P_{x\beta_{u_x}}^{k\,\text{b}} = 0
\end{aligned} \tag{50}$$

$$\begin{aligned}
\delta u_{y\alpha_{u_y}}^k :& -Z_{12\,u_y u_x}^{k\,\alpha_{u_y}\beta_{u_x}} u_{x\beta_{u_x},x,y}^k - Z_{22\,u_y u_y}^{k\,\alpha_{u_y}\beta_{u_y}} u_{y\beta_{u_y},y,y}^k - Z_{26\,u_y u_x}^{k\,\alpha_{u_y}\beta_{u_x}} u_{x\beta_{u_x},y,y}^k \\
& - Z_{26\,u_y u_y}^{k\,\alpha_{u_y}\beta_{u_y}} u_{y\beta_{u_y},x,y}^k - Z_{23\,u_y s_z}^{k\,\alpha_{u_y}\beta_{s_z}} s_{z\beta_{s_z},y}^k - Z_{16\,u_y u_x}^{k\,\alpha_{u_y}\beta_{u_x}} u_{x\beta_{u_x},x,x}^k - Z_{26\,u_y u_y}^{k\,\alpha_{u_y}\beta_{u_y}} u_{y\beta_{u_y},y,x}^k \\
& - Z_{66\,u_y u_x}^{k\,\alpha_{u_y}\beta_{u_x}} u_{x\beta_{u_x},y,x}^k - Z_{66\,u_y u_y}^{k\,\alpha_{u_y}\beta_{u_y}} u_{y\beta_{u_y},x,x}^k - Z_{36\,u_y s_z}^{k\,\alpha_{u_y}\beta_{s_z}} s_{z\beta_{s_z},x}^k + Z_{u_y s_y}^{k\,\alpha_{u_y},z\beta_{s_y}} s_{y\beta_{s_y}}^k \\
& - D_{u_y u_y}^{k\,t\,\alpha_{u_y}\beta_{u_y}} P_{y\beta_{u_y}}^{k\,\text{t}} - D_{u_y u_y}^{k\,b\,\alpha_{u_y}\beta_{u_y}} P_{y\beta_{u_y}}^{k\,\text{b}} = 0
\end{aligned} \tag{51}$$

$$\begin{aligned}
P\delta u_{z\alpha_{u_z}}^k :& -Z_{u_z s_x}^{k\,\alpha_{u_z}\beta_{s_x}} s_{x\beta_{s_x},x}^k - Z_{u_z s_y}^{k\,\alpha_{u_z}\beta_{s_y}} s_{y\beta_{s_y},y}^k + Z_{u_z s_z}^{k\,\alpha_{u_z},z\beta_{s_z}} s_{z\beta_{s_z}}^k - D_{u_z u_z}^{k\,t\,\alpha_{u_z}\beta_{u_z}} P_{z\beta_{u_z}}^{k\,\text{t}} \\
& - D_{u_z u_z}^{k\,b\,\alpha_{u_z}\beta_{u_z}} P_{z\beta_{u_z}}^{k\,\text{b}} = 0
\end{aligned} \tag{52}$$

$$\delta s_{x\alpha_{s_x}}^k : +Z_{s_x u_z}^{k\,\alpha_{s_x}\beta_{u_z}} u_{z\beta_{u_z},x}^k + Z_{s_x u_x}^{k\,\alpha_{s_x}\beta_{u_x},z} u_{x\beta_{u_x}}^k - Z_{55\,s_x s_x}^{k\,\alpha_{s_x}\beta_{s_x}} s_{x\beta_{s_x}}^k - Z_{45\,s_x s_y}^{k\,\alpha_{s_x}\beta_{s_y}} s_{y\beta_{s_y}}^k = 0 \tag{53}$$

$$\delta s_{y\alpha_{s_y}}^k : +Z_{s_y u_z}^{k\,\alpha_{s_y}\beta_{u_z}} u_{z\beta_{u_z},y}^k + Z_{s_y u_y}^{k\,\alpha_{s_y}\beta_{u_y},z} u_{y\beta_{u_y}}^k - Z_{45\,s_y s_x}^{k\,\alpha_{s_y}\beta_{s_x}} s_{x\beta_{s_x}}^k - Z_{44\,s_y s_y}^{k\,\alpha_{s_y}\beta_{s_y}} s_{y\beta_{s_y}}^k = 0 \tag{54}$$

$$\begin{aligned}
\delta s_{z\alpha_{s_z}}^k :& +Z_{s_z u_z}^{k\,\alpha_{s_z}\beta_{u_z},z} u_{z\beta_{u_z}}^k + Z_{13\,s_z u_x}^{k\,\alpha_{s_z}\beta_{u_x}} u_{x\beta_{u_x},x}^k + Z_{23\,s_z u_y}^{k\,\alpha_{s_z}\beta_{u_y}} u_{y\beta_{u_y},y}^k + Z_{36\,s_z u_x}^{k\,\alpha_{s_z}\beta_{u_x}} u_{x\beta_{u_x},y}^k \\
& + Z_{36\,s_z u_y}^{k\,\alpha_{s_z}\beta_{u_y}} u_{y\beta_{u_y},x}^k - Z_{33\,s_z s_z}^{k\,\alpha_{s_z}\beta_{s_z}} s_{z\beta_{s_z}}^k = 0
\end{aligned} \tag{55}$$

The boundary conditions are omitted for brevity.

## 8. Navier-type solution

Suppose that *only lamination schemes with angles 0 or 90 are used.* In this particular case, $\widetilde{C}_{16} = \widetilde{C}_{26} = \widetilde{C}_{36} = \widetilde{C}_{45} = 0$. From this relation, it is clear that the corresponding coefficients of the mixed form are zero as well: $C_{16} = C_{26} = C_{36} = C_{45} = 0$. Suppose also that the plate is a rectangular plate, where $a$ is the dimension in the $x$ direction, $b$ is the dimension in the $y$ direction. The external loads and displacements and stresses are assumed to have a sinusoidal distribution:

$$\begin{array}{ll} P^{k\,t}_{x\beta_{u_x}} = {}^xP^{k\,t}_{\beta_{u_x}} \mathcal{C}_a^{m\pi x}\mathcal{S}_b^{n\pi y} & P^{k\,b}_{x\beta_{u_x}} = {}^xP^{k\,b}_{\beta_{u_x}} \mathcal{C}_a^{m\pi x}\mathcal{S}_b^{n\pi y} \\ P^{k\,t}_{y\beta_{u_y}} = {}^yP^{k\,t}_{\beta_{u_y}} \mathcal{S}_a^{m\pi x}\mathcal{C}_b^{n\pi y} & P^{k\,b}_{y\beta_{u_y}} = {}^yP^{k\,b}_{\beta_{u_y}} \mathcal{S}_a^{m\pi x}\mathcal{C}_b^{n\pi y} \\ P^{k\,t}_{z\beta_{u_z}} = {}^zP^{k\,t}_{\beta_{u_z}} \mathcal{S}_a^{m\pi x}\mathcal{S}_b^{n\pi y} & P^{k\,b}_{z\beta_{u_z}} = {}^zP^{k\,b}_{\beta_{u_z}} \mathcal{S}_a^{m\pi x}\mathcal{S}_b^{n\pi y} \end{array} \tag{56}$$

$$\begin{array}{l} (u^k_{x\alpha_{u_x}}, s^k_{x\alpha_{s_x}}) = ({}^xU^k_{\alpha_{u_x}}, {}^xS^k_{\alpha_{s_x}})\mathcal{C}_a^{m\pi x}\mathcal{S}_b^{n\pi y} \\ (u^k_{y\alpha_{u_y}}, s^k_{y\alpha_{s_y}}) = ({}^yU^k_{\alpha_{u_y}}, {}^yS^k_{\alpha_{s_y}})\mathcal{S}_a^{m\pi x}\mathcal{C}_b^{n\pi y} \\ (u^k_{z\alpha_{u_z}}, s^k_{z\alpha_{s_z}}) = ({}^zU^k_{\alpha_{u_z}}, {}^zS^k_{\alpha_{s_z}})\mathcal{S}_a^{m\pi x}\mathcal{S}_b^{n\pi y} \end{array} \tag{57}$$

where the following definitions have been used:

$$\begin{array}{ll} \mathcal{C}_a^{m\pi x} = \cos\dfrac{m\pi x}{a} & \mathcal{S}_b^{n\pi y} = \sin\dfrac{n\pi y}{b} \\ \mathcal{S}_a^{m\pi x} = \sin\dfrac{m\pi x}{a} & \mathcal{C}_b^{n\pi y} = \cos\dfrac{n\pi y}{b} \end{array} \tag{58}$$

The assumptions used for the displacements and transverse stresses by using trigonometric functions solve exactly the problem of simply supported plate [41]. With these assumptions, the governing equations become

$$\begin{aligned} \delta u_{x\alpha_{u_x}} &: +K^{k\,\alpha_{u_x}\beta_{u_x}}_{u_x u_x}\,{}^xU^k_{\beta_{u_x}} + K^{k\,\alpha_{u_x}\beta_{u_y}}_{u_x u_y}\,{}^yU^k_{\beta_{u_y}} + K^{k\,\alpha_{u_x}\beta_{s_x}}_{u_x s_x}\,{}^xS^k_{\beta_{s_x}} \\ &\quad + K^{k\,\alpha_{u_x}\beta_{s_z}}_{u_x s_z}\,{}^zS^k_{\beta_{s_z}} = {}^xR^k_{\alpha_{u_x}} \\ \delta u_{y\alpha_{u_y}} &: +K^{k\,\alpha_{u_y}\beta_{u_x}}_{u_y u_x}\,{}^xU^k_{\beta_{u_x}} + K^{k\,\alpha_{u_y}\beta_{u_y}}_{u_y u_y}\,{}^yU^k_{\beta_{u_y}} + K^{k\,\alpha_{u_y}\beta_{s_y}}_{u_y s_y}\,{}^yS^k_{\beta_{s_y}} \\ &\quad + K^{k\,\alpha_{u_y}\beta_{s_z}}_{u_y s_z}\,{}^zS^k_{\beta_{s_z}} = {}^yR^k_{\alpha_{u_y}} \\ \delta u_{z\alpha_{u_z}} &: +K^{k\,\alpha_{u_z}\beta_{s_x}}_{u_z s_x}\,{}^xS^k_{\beta_{s_x}} + K^{k\,\alpha_{u_z}\beta_{s_y}}_{u_z s_y}\,{}^yS^k_{\beta_{s_y}} + K^{k\,\alpha_{u_z}\beta_{s_z}}_{u_z s_z}\,{}^zS^k_{\beta_{s_z}} = {}^zR^k_{\alpha_{u_z}} \\ \delta s_{x\alpha_{s_x}} &: +K^{k\,\alpha_{s_x}\beta_{u_x}}_{s_x u_x}\,{}^xU^k_{\beta_{u_x}} + K^{k\,\alpha_{s_x}\beta_{u_z}}_{s_x u_z}\,{}^zU^k_{\beta_{u_z}} + K^{k\,\alpha_{s_x}\beta_{s_x}}_{s_x s_x}\,{}^xS^k_{\beta_{s_x}} = 0 \\ \delta s_{y\alpha_{s_y}} &: +K^{k\,\alpha_{s_y}\beta_{u_y}}_{s_y u_y}\,{}^yU^k_{\beta_{u_y}} + K^{k\,\alpha_{s_y}\beta_{u_z}}_{s_y u_z}\,{}^zU^k_{\beta_{u_z}} + K^{k\,\alpha_{s_y}\beta_{s_y}}_{s_y s_y}\,{}^yS^k_{\beta_{s_y}} = 0 \\ \delta s_{z\alpha_{s_z}} &: +K^{k\,\alpha_{s_z}\beta_{u_x}}_{s_z u_x}\,{}^xU^k_{\beta_{u_x}} + K^{k\,\alpha_{s_z}\beta_{u_y}}_{s_z u_y}\,{}^yU^k_{\beta_{u_y}} + K^{k\,\alpha_{s_z}\beta_{u_z}}_{s_z u_z}\,{}^zU^k_{\beta_{u_z}} + K^{k\,\alpha_{s_z}\beta_{s_z}}_{s_z s_z}\,{}^zS^k_{\beta_{s_z}} = 0 \end{aligned} \tag{59}$$

where the *loads* have been *defined* as follows:

$$\begin{aligned} {}^xR^k_{\alpha_{u_x}} &= D^{k\,t\,\alpha_{u_x}\beta_{u_x}}_{u_x u_x}\;{}^xP^{k\,t}_{\beta_{u_x}} + D^{k\,b\,\alpha_{u_x}\beta_{u_x}}_{u_x u_x}\;{}^xP^{k\,b}_{\beta_{u_x}} \\ {}^yR^k_{\alpha_{u_y}} &= D^{k\,t\,\alpha_{u_y}\beta_{u_y}}_{u_y u_y}\;{}^yP^{k\,t}_{\beta_{u_y}} + D^{k\,b\,\alpha_{u_y}\beta_{u_y}}_{u_y u_y}\;{}^yP^{k\,b}_{\beta_{u_y}} \\ {}^zR^k_{\alpha_{u_z}} &= D^{k\,t\,\alpha_{u_z}\beta_{u_z}}_{u_z u_z}\;{}^zP^{k\,t}_{\beta_{u_z}} + D^{k\,b\,\alpha_{u_z}\beta_{u_z}}_{u_z u_z}\;{}^zP^{k\,b}_{\beta_{u_z}} \end{aligned} \tag{60}$$

The 21 fundamental nuclei or *kernels of the generalized unified formulation* (see Eq. 59) are defined as follows:

$$\begin{aligned} K^{k\,\alpha_{u_x}\beta_{u_x}}_{u_x u_x} &= \frac{m^2\pi^2}{a^2} Z^{k\,\alpha_{u_x}\beta_{u_x}}_{11\;u_x u_x} + \frac{n^2\pi^2}{b^2} Z^{k\,\alpha_{u_x}\beta_{u_x}}_{66\;u_x u_x} \\ K^{k\,\alpha_{u_x}\beta_{u_y}}_{u_x u_y} &= \frac{mn\pi^2}{ab} Z^{k\,\alpha_{u_x}\beta_{u_y}}_{12\;u_x u_y} + \frac{mn\pi^2}{ab} Z^{k\,\alpha_{u_x}\beta_{u_y}}_{66\;u_x u_y} \\ K^{k\,\alpha_{u_x}\beta_{s_z}}_{u_x s_z} &= -\frac{m\pi}{a} Z^{k\,\alpha_{u_x}\beta_{s_z}}_{13\;u_x s_z}; \quad K^{k\,\alpha_{u_x}\beta_{s_x}}_{u_x s_x} = +Z^{k\,\alpha_{u_x,z}\beta_{s_x}}_{u_x s_x} \\ K^{k\,\alpha_{u_y}\beta_{u_x}}_{u_y u_x} &= +\frac{mn\pi^2}{ab} Z^{k\,\alpha_{u_y}\beta_{u_x}}_{12\;u_y u_x} + \frac{mn\pi^2}{ab} Z^{k\,\alpha_{u_y}\beta_{u_x}}_{66\;u_y u_x} \\ K^{k\,\alpha_{u_y}\beta_{u_y}}_{u_y u_y} &= +\frac{n^2\pi^2}{b^2} Z^{k\,\alpha_{u_y}\beta_{u_y}}_{22\;u_y u_y} + \frac{m^2\pi^2}{a^2} Z^{k\,\alpha_{u_y}\beta_{u_y}}_{66\;u_y u_y} \\ K^{k\,\alpha_{u_y}\beta_{s_z}}_{u_y s_z} &= -\frac{n\pi}{b} Z^{k\,\alpha_{u_y}\beta_{s_z}}_{23\;u_y s_z}; \quad K^{k\,\alpha_{u_y}\beta_{s_y}}_{u_y s_y} = +Z^{k\,\alpha_{u_y,z}\beta_{s_y}}_{u_y s_y} \\ K^{k\,\alpha_{u_z}\beta_{s_x}}_{u_z s_x} &= \frac{m\pi}{a} Z^{k\,\alpha_{u_z}\beta_{s_x}}_{u_z s_x}; \quad K^{k\,\alpha_{u_z}\beta_{s_y}}_{u_z s_y} = +\frac{n\pi}{b} Z^{k\,\alpha_{u_z}\beta_{s_y}}_{u_z s_y} \\ K^{k\,\alpha_{u_z}\beta_{s_z}}_{u_z s_z} &= +Z^{k\,\alpha_{u_z,z}\beta_{s_z}}_{u_z s_z}; \quad K^{k\,\alpha_{s_x}\beta_{u_z}}_{s_x u_z} = \frac{m\pi}{a} Z^{k\,\alpha_{s_x}\beta_{u_z}}_{s_x u_z} \\ K^{k\,\alpha_{s_x}\beta_{u_x}}_{s_x u_x} &= Z^{k\,\alpha_{s_x}\beta_{u_x,z}}_{s_x u_x}; \quad K^{k\,\alpha_{s_x}\beta_{s_x}}_{s_x s_x} = -Z^{k\,\alpha_{s_x}\beta_{s_x}}_{55\;s_x s_x} \\ K^{k\,\alpha_{s_y}\beta_{u_z}}_{s_y u_z} &= +\frac{n\pi}{b} Z^{k\,\alpha_{s_y}\beta_{u_z}}_{s_y u_z}; \quad K^{k\,\alpha_{s_y}\beta_{u_y}}_{s_y u_y} = Z^{k\,\alpha_{s_y}\beta_{u_y,z}}_{s_y u_y} \\ K^{k\,\alpha_{s_y}\beta_{s_y}}_{s_y s_y} &= -Z^{k\,\alpha_{s_y}\beta_{s_y}}_{44\;s_y s_y}; \quad K^{k\,\alpha_{s_z}\beta_{u_z}}_{s_z u_z} = Z^{k\,\alpha_{s_z}\beta_{u_z,z}}_{s_z u_z} \\ K^{k\,\alpha_{s_z}\beta_{u_x}}_{s_z u_x} &= -\frac{m\pi}{a} Z^{k\,\alpha_{s_z}\beta_{u_x}}_{13\;s_z u_x}; \quad K^{k\,\alpha_{s_z}\beta_{u_y}}_{s_z u_y} = -\frac{n\pi}{b} Z^{k\,\alpha_{s_z}\beta_{u_y}}_{23\;s_z u_y} \\ K^{k\,\alpha_{s_z}\beta_{s_z}}_{s_z s_z} &= -Z^{k\,\alpha_{s_z}\beta_{s_z}}_{33\;s_z s_z} \end{aligned} \tag{61}$$

At this point, equivalent single layer models (with or without Murakami's zig-zag function) and layerwise models have the same formal expressions (this concepts will be clarified later). The distinction between the different types of theories presented is in the assembling. *Part II* (see [37]) will analyze the case of layerwise models, *Part III* (see [38]) will analyze the mixed higher order shear deformation theories and *Part IV* (see [39]) will consider the case of mixed zig-zag plate theories.

Infinite theories can be obtained by expanding the kernels of Eq. (61). In particular, the *matrices at layer level* are obtained by expanding the $1 \times 1$ kernels. Then, the layer matrices are assembled in the thickness direction according to the category of the analyzed theories (see *Part II*, *Part III* and *Part IV*). Suppose that the generation of the matrices and assembling process for a particular theory (it is not relevant to specify if the theory is an equivalent single layer or a layerwise theory and what orders are used for the expansions of the different variables) have been performed. The governing equations are written as follows:

$$\begin{bmatrix} \mathbf{K}_{u_x u_x} & \mathbf{K}_{u_x u_y} & \mathbf{0}_{u_x u_z} & \mathbf{K}_{u_x s_x} & \mathbf{0}_{u_x s_y} & \mathbf{K}_{u_x s_z} \\ & \mathbf{K}_{u_y u_y} & \mathbf{0}_{u_y u_z} & \mathbf{0}_{u_y s_x} & \mathbf{K}_{u_y s_y} & \mathbf{K}_{u_y s_z} \\ & & \mathbf{0}_{u_z u_z} & \mathbf{K}_{u_z s_x} & \mathbf{K}_{u_z s_y} & \mathbf{K}_{u_z s_z} \\ & & & \mathbf{K}_{s_x s_x} & \mathbf{0}_{s_x s_y} & \mathbf{0}_{s_x s_z} \\ \text{Symm} & & & & \mathbf{K}_{s_y s_y} & \mathbf{0}_{s_y s_z} \\ & & & & & \mathbf{K}_{s_z s_z} \end{bmatrix} \cdot \begin{bmatrix} {}^x\mathbf{U} \\ {}^y\mathbf{U} \\ {}^z\mathbf{U} \\ {}^x\mathbf{S} \\ {}^y\mathbf{S} \\ {}^z\mathbf{S} \end{bmatrix} = \begin{bmatrix} {}^x\mathbf{R} \\ {}^y\mathbf{R} \\ {}^z\mathbf{R} \\ {}^x\mathbf{0} \\ {}^y\mathbf{0} \\ {}^z\mathbf{0} \end{bmatrix} \tag{62}$$

From Eq. (62) it can also be deduced that only 13 kernels (see Eq. 61) are really required for the generation of *any* theory. IN FEM applications it is possible to demonstrate that in general the required kernels are 14 and not 13. In particular, the kernel $K^{k\,\alpha_{s_x}\beta_{s_y}}_{s_x s_y}$ is different than zero and has to be considered in the generation of the theories. In the present Navier Type solution that kernel is zero because the lamination scheme can be only with angles of zero or 90 degrees.

In order to eliminate the stress variables (Static Condensation Technique), it is convenient to partition Eq. (62) as follows:

$$\begin{cases} \mathbf{K}_{UU}\mathbf{U} + \mathbf{K}_{US}\mathbf{S} = \mathbf{R} \\ \mathbf{K}_{SU}\mathbf{U} + \mathbf{K}_{SS}\mathbf{S} = \mathbf{0} \end{cases} \tag{63}$$

Substituting the second equation into the first equation:

$$\begin{cases} \mathbf{S} = -(\mathbf{K}_{SS})^{-1}\mathbf{K}_{SU}\mathbf{U} \\ \mathbf{K}_{UU}\mathbf{U} - \mathbf{K}_{US}(\mathbf{K}_{SS})^{-1}\mathbf{K}_{SU}\mathbf{U} = \mathbf{R} \end{cases} \tag{64}$$

The first equation of system 64 is used to calculate the stresses a posteriori and the second equation of system 64 becomes

$$(\mathbf{K}_{UU} - \mathbf{K}_{US}(\mathbf{K}_{SS})^{-1}\mathbf{K}_{SU})\mathbf{U} = \mathbf{R} \Rightarrow \mathbf{K}_{\text{mixed}}\mathbf{U} = \mathbf{R} \tag{65}$$

The matrix $\mathbf{K}_{\text{mixed}}$ can be rendered explicit and, thus, Eq. (65) becomes:

$$\begin{bmatrix} \overline{\mathbf{K}}_{u_x u_x} & \overline{\mathbf{K}}_{u_x u_y} & \overline{\mathbf{K}}_{u_x u_z} \\ \overline{\mathbf{K}}_{u_y u_x} & \overline{\mathbf{K}}_{u_y u_y} & \overline{\mathbf{K}}_{u_y u_z} \\ \overline{\mathbf{K}}_{u_z u_x} & \overline{\mathbf{K}}_{u_z u_y} & \overline{\mathbf{K}}_{u_z u_z} \end{bmatrix} \cdot \begin{bmatrix} {}^x\mathbf{U} \\ {}^y\mathbf{U} \\ {}^z\mathbf{U} \end{bmatrix} = \begin{bmatrix} {}^x\mathbf{R} \\ {}^y\mathbf{R} \\ {}^z\mathbf{R} \end{bmatrix} \tag{66}$$

Notice that $\overline{\mathbf{K}}_{u_x u_x} \neq \mathbf{K}_{u_x u_x}$ and so forth. In the case of Navier-type solution the static condensation technique (SCT) produces exactly the same results as the case in which SCT is not performed. This is because the condensation is performed at structural level. In FEM application, the static condensation is conveniently applied at element level to formally obtain a displacement-based formulation. However, the continuity of the out-of-plane stresses in the plane is not enforced and, thus, the static condensation technique differs in this case from the "full" case.

Once the static condensation has been performed and the amplitudes of the displacements are calculated, the amplitudes of the stresses can be calculated using the first relation of Eq. (64).

## 9. Conclusion

The first part of this work presented the governing equations for $\infty^6$ mixed theories based on Reissner's mixed variational theorem. This formal technique allows the generation of a very large variety of theories: layerwise models, mixed higher order shear deformation theories and zig-zag theories which can *all* be generated from only thirteen *kernels of the generalized unified formulation.* The kernels are $1 \times 1$ matrices and have the fundamental property of being invariant with respect to the type of theory (e.g., layerwise or mixed higher order shear deformation theories) and the orders used in the expansion of the different variables. For example, the layerwise theory with cubic order for the in-plane displacements $u_x, u_y$ and stresses $\sigma_{zx}, \sigma_{zy}$ and parabolic order for the displacement $u_z$ and stress $\sigma_{zz}$ and the mixed higher order theory with parabolic order for the displacements $u_x, u_y, u_z$ and stresses $\sigma_{zx}, \sigma_{zy}, \sigma_{zz}$ *have the same kernels.* This fundamental property allows the creation of a powerful tool, which can analyze plates with very high accuracy (layerwise models with high orders used for the displacements and stresses) or with low computational cost if the case is "less challenging" such as the case of isotropic thin plates. The generalized unified formulation also allows an independent treatment of the unknowns, and this could be used for the generation of new numerical approaches in FEM applications. *Part II* of this work will analyze the case of layerwise theories. *Part III* will focus on the case of higher order theories and *Part IV* will discuss the zig-zag theories. Finally, *Part V* will show numerous comparisons between some of the infinite possible theories.

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