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∞^6 Mixed plate theories based on the Generalized Unified Formulation Part II: Layerwise theories

Luciano Demasi *

Department of Aerospace Engineering and Engineering Mechanics, San Diego State University, 5500 Campanile Drive, San Diego, CA, USA

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ABSTRACT

The generalized unified formulation introduced in *Part I* for the case of composite plates and Reissner's Mixed variational theorem is, for the first time in the literature, applied to the case of layerwise theories. Each layer is independently modeled. The compatibility of the displacements and the equilibrium of the transverse stresses between two adjacent layers are enforced a priori. Infinite combinations of the orders used for displacements u_x, u_y, u_z and out-of-plane stresses σ_{zx}, σ_{zy}, σ_{zz} can be freely chosen. ∞^6 layerwise theories are therefore presented. The code based on this capability can have all the possible ∞^6 theories built-in, thus, making the code a powerful and versatile tool to analyze different geometries, boundary conditions and applied loads. *All* ∞^6 theories are generated by expanding 13 1×1 *invariant* matrices (the kernels of the Generalized Unified Formulation). How the kernels are expanded and the theories generated is explained. Details of the assembling in the thickness direction and the generation of the matrices are provided.

1. Introduction

1.1. Layerwise models: main theoretical concepts

Equivalent single layer theories give a sufficiently accurate description of the global laminate response. However, these theories are not adequate for determining the stress fields at ply level. Layerwise theories assume separate displacement field expansions within each layer. The accuracy is then greater but the price is in the increased computational cost. Many layerwise plate models have been proposed in the past by applying classical plate theory or higher order theories at each layer. Generalizations of these approaches were also given, and the displacements variables were expressed in terms of Lagrange polynomials. Among the papers devoted on the subject of layerwise theories, Refs. [1–5] give an idea of the different approaches. Normally, displacement-based layerwise models do not a priori take into account the continuity of the transverse stresses between two adjacent layers. The problem of satisfying the interlaminar continuity of the transverse stresses a priori led to the derivation of mixed layerwise theories [6–8]. Layerwise theories could also be easily extended to the case of composite beams as explained in Ref. [9]. The conceptual differences between the displacement fields in layerwise and equivalent single layer theories are depicted in Fig. 1. Layerwise models are computationally more expensive than the less accurate equivalent single layer models. Therefore, layerwise models can be used in regions of the structure in which an accurate description is required [10], whereas equivalent single layer models are employed in other parts of the structure. It is also possible to develop quasi-layerwise theories in which some quantities are described using the layerwise approach and some are described using the equivalent single layer approach. These two categories of theories will be described in *Part III* Ref. [11] and *Part IV* Ref. [12] of this work.

1.2. What are the new contributions of this work

The generalized unified formulation (GUF) [13,14] is a new formalism and a generalization of Carrera's Unified Formulation (CUF) [15]. GUF was introduced in the case of displacement-based theories. GUF was extended, for the first time in the literature, to the case of mixed theories (see *Part I*, Ref. [16]). In particular, Reissner's mixed variational theorem (RMVT) (see [17,18]) was employed. The unknown variables were the displacements and transverse stresses.

In the present work GUF will be extended to the case of composite multilayered structures analyzed with *layerwise* models. Each layer variable (a displacement or a transverse stress) will be *independently* expanded along the thickness leading to a very wide variety of new theories. Since each variable can be expanded in infinite different forms (by simply changing the order of the polynomial used in the expansion along the thickness), the case of RMVT-based theories leads to the writing of ∞^6 layerwise mixed theories. *All* the possible theories generated using GUF can be

* Tel.: +1 (619) 594 3752; fax: +1 619 594 6005.
E-mail address: ldemasi@mail.sdsu.edu

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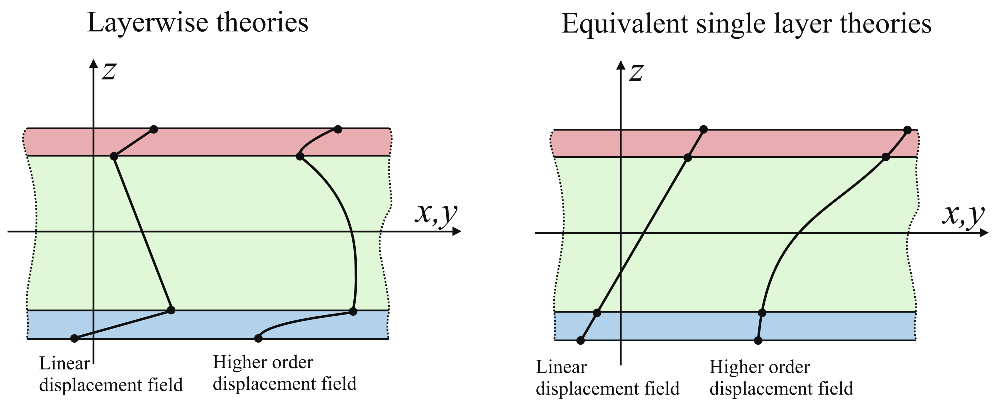

Fig. 1. Layerwise theories vs equivalent single layer theories in a three layered structure.

implemented in a single code without the requirements of new implementations or theoretical developments.

This is a new powerful methodology to create layerwise theories, and the details will be given in this work. In particular, it will be shown how to generate the layer matrices from the fundamental nuclei or kernels (introduced in *Part I*, Ref. [16]) and how to assemble these matrices. The interlaminar continuity of the displacements and transverse stresses is taken into account. Numerous examples clarify how to use GUF to generate a desired layerwise theory.

2. Theoretical derivation of ∞^6 layerwise mixed theories

For a generic layer k, the displacements u_x^k, u_y^k, u_z^k and out-of-plane stresses $\sigma_{xz}^k = s_x^k$, $\sigma_{yz}^k = s_y^k$, $\sigma_{zz}^k = s_z^k$ are written in a compact notation (the generalized unified formulation) as follows:

$$
\begin{aligned}
& u_x^k = {}^xF_t u_{x_t}^k + {}^xF_l u_{x_l}^k + {}^xF_b u_{x_b}^k = {}^xF_{\alpha_{u_x}} u_{x\alpha_{u_x}}^k \\
& \alpha_{u_x} = t, l, b; \quad l = 2, \ldots, N_{u_x} \\
& u_y^k = {}^yF_t u_{y_t}^k + {}^yF_m u_{y_m}^k + {}^yF_b u_{y_b}^k = {}^yF_{\alpha_{u_y}} u_{y\alpha_{u_y}}^k \\
& \alpha_{u_y} = t, m, b; \quad m = 2, \ldots, N_{u_y} \\
& u_z^k = {}^zF_t u_{z_t}^k + {}^zF_n u_{z_n}^k + {}^zF_b u_{z_b}^k = {}^zF_{\alpha_{u_z}} u_{z\alpha_{u_z}}^k \\
& \alpha_{u_z} = t, n, b; \quad n = 2, \ldots, N_{u_z} \\
& s_x^k = {}^x\mathcal{F}_t s_{x_t}^k + {}^x\mathcal{F}_p s_{x_p}^k + {}^x\mathcal{F}_b s_{x_b}^k = {}^x\mathcal{F}_{\alpha_{s_x}} s_{x\alpha_{s_x}}^k \\
& \alpha_{s_x} = t, p, b; \quad p = 2, \ldots, N_{s_x} \\
& s_y^k = {}^y\mathcal{F}_t s_{y_t}^k + {}^y\mathcal{F}_q s_{y_q}^k + {}^y\mathcal{F}_b s_{y_b}^k = {}^y\mathcal{F}_{\alpha_{s_y}} s_{y\alpha_{s_y}}^k \\
& \alpha_{s_y} = t, q, b; \quad q = 2, \ldots, N_{s_y} \\
& s_z^k = {}^z\mathcal{F}_t s_{z_t}^k + {}^z\mathcal{F}_r s_{z_r}^k + {}^z\mathcal{F}_b s_{z_b}^k = {}^z\mathcal{F}_{\alpha_{s_z}} s_{z\alpha_{s_z}}^k \\
& \alpha_{s_z} = t, r, b; \quad r = 2, \ldots, N_{s_z}
\end{aligned} \tag{1}
$$

The functions of the thickness coordinate are introduced in a general form. For example, ${}^xF_t$ is a function of z and can be a polynomial, trigonometric, exponential or another function chosen a priori. To have the assembling process along the thickness direction immediate and intuitive and indicated for the case of multilayered structures a *convenient* expansion along the thickness is introduced.

The displacements and out-of-plane stresses must be continuous functions along the thickness to ensure the compatibility of the displacements and the equilibrium between two adjacent layers. Therefore, it is convenient for the axiomatic expansions along the thickness to have the following properties:

- *Property 1*
  For $z = z_{\text{bot}_k}$ (the bottom surface of layer k; see Fig. 1 in *Part I*, Ref. [16]) all the functions along the thickness are zero except the one which multiplies the term corresponding to the bottom (subscript b). For example, in the case of the displacement u_x^k, the functions calculated at the bottom of layer k should give the following values:

$$
{}^xF_t(z = z_{\text{bot}_k}) = 0 \qquad {}^xF_l(z = z_{\text{bot}_k}) = 0 \qquad {}^xF_b(z = z_{\text{bot}_k}) = 1 \tag{2}
$$

If the previous conditions are satisfied then from the first expression of Eq. (1) it is possible to deduce:

$$
u_x^k(z = z_{\text{bot}_k}) = 0 \cdot u_{x_t}^k + 0 \cdot u_{x_l}^k + 1 \cdot u_{x_b}^k = u_{x_b}^k \tag{3}
$$

Therefore, *if* the conditions reported in Eq. (2) are satisfied, $u_{x_b}^k$ is not just a term in the thickness expansion of the variable u_x^k but assumes the meaning of the value that the displacement u_x^k takes when the bottom surface of layer k is considered (i.e., $z = z_{\text{bot}_k}$). This now explains why the subscript "b" is introduced in the notation.

- *Property 2*
  For $z = z_{\text{top}_k}$ (the top surface of layer k) all the functions along the thickness are zero except the one which multiplies the term corresponding to the top (subscript t). For example, in the case of the displacement u_x^k, the functions calculated at the top of layer k should give the following values:

$$
{}^xF_t(z = z_{\text{top}_k}) = 1 \qquad {}^xF_l(z = z_{\text{top}_k}) = 0 \qquad {}^xF_b(z = z_{\text{top}_k}) = 0 \tag{4}
$$

If the previous conditions are satisfied then from the first expression of Eq. (1) it is possible to obtain:

$$
u_x^k(z = z_{\text{top}_k}) = 1 \cdot u_{x_t}^k + 0 \cdot u_{x_l}^k + 0 \cdot u_{x_b}^k = u_{x_t}^k \tag{5}
$$

Therefore, *if* the conditions reported in Eq. (4) are satisfied, $u_{x_t}^k$ is not just a term in the thickness expansion of the variable u_x^k but assumes the meaning of the value that the displacement u_x^k takes when the top surface of layer k is considered (i.e., $z = z_{\text{top}_k}$). This now explains why the subscript "t" is introduced.

- *Property 3*
  It is known that polynomial functions of the type z_k^n are responsible for ill conditioning (see a discussion of this problem in [19]) when n is increased. This can be avoided by using *orthogonal polynomials.*

A good set of functions (for all the displacements and out-of-plane stresses) which satisfy the above mentioned properties should be selected. It is possible to demonstrate that all the previous properties are satisfied if particular combination of Legendre

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polynomials is used. Legendre polynomials are defined in the interval $[-1, 1]$. Thus, a transformation is necessary:

$$\zeta_k = \frac{2}{z_{\mathrm{top}_k} - z_{\mathrm{bot}_k}} z - \frac{z_{\mathrm{top}_k} + z_{\mathrm{bot}_k}}{z_{\mathrm{top}_k} - z_{\mathrm{bot}_k}} \qquad -1 \leqslant \zeta_k \leqslant +1 \tag{6}$$

where ζ_k is a non-dimensional coordinate. The following formula is also valid:

$$z_{\mathrm{top}_k} = z_{\mathrm{bot}_k} + h_k \tag{7}$$

The transformation is then

$$\zeta_k = \frac{2}{h_k} z - \frac{z_{\mathrm{top}_k} + z_{\mathrm{bot}_k}}{z_{\mathrm{top}_k} - z_{\mathrm{bot}_k}} \tag{8}$$

The Legendre polynomial of order zero is $P_0(\zeta_k) = 1$. The Legendre polynomial of order one is $P_1(\zeta_k) = \zeta_k$. The higher order polynomials can be obtained by using Bonnet's recursion [20]:

$$P_{n+1}(\zeta_k) = \frac{(2n+1)\zeta_k P_n(\zeta_k) - nP_{n-1}(\zeta_k)}{n+1} \tag{9}$$

Bonnet's formula is a convenient method to calculate the Legendre polynomials in a practical code based on the generalized unified formulation.

The explicit form of the Legendre polynomials of order 2, 3, 4 and 5 are the following (but in practice these formulas are not convenient and the recursive method previously introduced should be used):

$$\begin{array}{ll} P_2(\zeta_k) = \frac{3(\zeta_k)^2 - 1}{2} & P_3(\zeta_k) = \frac{5(\zeta_k)^3 - 3\zeta_k}{2} \\ P_4(\zeta_k) = \frac{35(\zeta_k)^4 - 30(\zeta_k)^2 + 3}{8} & P_5(\zeta_k) = \frac{63(\zeta_k)^5 - 70(\zeta_k)^3 + 15\zeta_k}{8} \end{array} \tag{10}$$

The same functions for all displacements are used. This is not necessary with the generalized unified formulation but it is more practical. The following combination of Legendre functions is used:

$$\begin{array}{l} {}^xF_t = {}^yF_t = {}^zF_t = \frac{P_0+P_1}{2}, {}^xF_b = {}^yF_b = {}^zF_b = \frac{P_0-P_1}{2} \\ {}^xF_l = P_l - P_{l-2}, l = 2, 3, \ldots, N_{u_x} \\ {}^yF_m = P_m - P_{m-2}, m = 2, 3, \ldots, N_{u_y} \\ {}^zF_n = P_n - P_{n-2}, n = 2, 3, \ldots, N_{u_z} \end{array} \tag{11}$$

in which $P_j = P_j(\zeta_k)$ is the Legendre polynomial of j-order. The chosen functions have the following properties:

$$\zeta_k = \begin{cases} +1, & {}^xF_t, {}^yF_t, {}^zF_t = 1, {}^xF_b, {}^yF_b, {}^zF_b = 0, {}^xF_l, {}^yF_m, {}^zF_n = 0 \\ -1, & {}^xF_t, {}^yF_t, {}^zF_t = 0, {}^xF_b, {}^yF_b, {}^zF_b = 1, {}^xF_l, {}^yF_m, {}^zF_n = 0 \end{cases} \tag{12}$$

Thus, the properties earlier mentioned are all satisfied and this set of functions is a good choice to build the mixed layerwise theories. It is convenient (but not necessary) to use the same type of functions for the thickness expansions of the stresses:

$$\begin{array}{l} {}^x\mathcal{F}_t = {}^y\mathcal{F}_t = {}^z\mathcal{F}_t = \frac{P_0+P_1}{2}, {}^x\mathcal{F}_b = {}^y\mathcal{F}_b = {}^z\mathcal{F}_b = \frac{P_0-P_1}{2} \\ {}^x\mathcal{F}_p = P_p - P_{p-2}, \quad p = 2, 3, \ldots, N_{s_x} \\ {}^y\mathcal{F}_q = P_q - P_{q-2}, \quad q = 2, 3, \ldots, N_{s_y} \\ {}^z\mathcal{F}_r = P_r - P_{r-2}, \quad r = 2, 3, \ldots, N_{s_z} \end{array} \tag{13}$$

With the generalized unified formulation other functions could be used without changing the formalism. However, combination of Legendre's polynomials has been proven effective and convenient (see Ref. [6]). It is then possible to create a class of theories by changing the orders of displacements and stresses. Suppose, for example, that a theory has the following data: $N_{u_x} = 3$, $N_{u_y} = 2$, $N_{u_z} = 4$, $N_{s_x} = 5$, $N_{s_y} = 4$, $N_{s_z} = 6$. The corresponding theory is indicated as LM_{324}^{546}. The first letter "L" means "Layerwise" theory, the second letter "M" means that a "mixed" variational theorem is used (Reissner's variational theorem). The subscripts are the orders of the Legendre polynomials used for the displacements. The superscripts are the orders of the Legendre polynomials used for the out-of-plane stresses. In general, the acronym is then built as follows: $\mathrm{LM}_{N_{u_x} N_{u_y} N_{u_z}}^{N_{s_x} N_{s_y} N_{s_z}}$.

Mixed theories based on RMVT and a Compact Notation were also introduced by Carrera [6–8]. In particular, he formulated the problem with Carrera's Unified Formulation (see the discussions in [13,15]). The main differences between CUF and GUF and the notations for layerwise theories are shown in Fig. 2. If the Static Condensation Technique (SCT) *is not* performed then the theory is indicated with the acronym $\mathrm{LMF}_{N_{u_x} N_{u_y} N_{u_z}}^{N_{s_x} N_{s_y} N_{s_z}}$, where "F" means "full". The other quantities have the same meaning as before.

A more detailed discussion on the static condensation technique is reported in another of author's work [21]. In particular, FEM applications of Carrera's Unified Formulation are analyzed, and it is shown that SCT is fundamental to reduce the CPU time for the (already computationally expensive) layerwise theories. In fact, using the "full" approach improves the calculation of the out-of-plane stresses but the price is too high. Therefore, for the common engineering applications of mixed layerwise theories SCT should be performed at finite element level.

3. How ∞^6 layerwise theories are generated

3.1. Kernels of the generalized unified formulation

In *Part I* Ref. [16] the governing equations (Navier-type solution) were written with a Compact Notation: the generalized unified formulation. All of the equations, including ∞^6 different combinations, were written as function of *kernels of the generalized unified formulation*. In particular, six pressure kernels of 1×1 matrices were introduced. Also, 21 kernels were used to generate the matrices (but only 13 are really required). The fundamental equations, *invariant* with respect to the orders used for the expansions of the different variables, are the following:

$$\begin{array}{l}
\overbrace{K_{u_x u_x}^{k\alpha_{u_x}\beta_{u_x}}}^{1\times1\,\text{kernel}} \overbrace{{}^xU_{\beta_{u_x}}^k}^{\text{unknown}} + \overbrace{K_{u_x u_y}^{k\alpha_{u_x}\beta_{u_y}}}^{1\times1\,\text{kernel}} \overbrace{{}^yU_{\beta_{u_y}}^k}^{\text{unknown}} + \overbrace{K_{u_x s_x}^{k\alpha_{u_x}\beta_{s_x}}}^{1\times1\,\text{kernel}} \overbrace{{}^xS_{\beta_{s_x}}^k}^{\text{unknown}} + \overbrace{K_{u_x s_z}^{k\alpha_{u_x}\beta_{s_z}}}^{1\times1\,\text{kernel}} \overbrace{{}^zS_{\beta_{s_z}}^k}^{\text{unknown}} = \overbrace{{}^xR_{\alpha_{u_x}}^k}^{\text{known}} \\
\overbrace{K_{u_y u_x}^{k\alpha_{u_y}\beta_{u_x}}}^{1\times1\,\text{kernel}} \overbrace{{}^xU_{\beta_{u_x}}^k}^{\text{unknown}} + \overbrace{K_{u_y u_y}^{k\alpha_{u_y}\beta_{u_y}}}^{1\times1\,\text{kernel}} \overbrace{{}^yU_{\beta_{u_y}}^k}^{\text{unknown}} + \overbrace{K_{u_y s_y}^{k\alpha_{u_y}\beta_{s_y}}}^{1\times1\,\text{kernel}} \overbrace{{}^yS_{\beta_{s_y}}^k}^{\text{unknown}} + \overbrace{K_{u_y s_z}^{k\alpha_{u_y}\beta_{s_z}}}^{1\times1\,\text{kernel}} \overbrace{{}^zS_{\beta_{s_z}}^k}^{\text{unknown}} = \overbrace{{}^yR_{\alpha_{u_y}}^k}^{\text{known}} \\
\overbrace{K_{u_z s_x}^{k\alpha_{u_z}\beta_{s_x}}}^{1\times1\,\text{kernel}} \overbrace{{}^xS_{\beta_{s_x}}^k}^{\text{unknown}} + \overbrace{K_{u_z s_y}^{k\alpha_{u_z}\beta_{s_y}}}^{1\times1\,\text{kernel}} \overbrace{{}^yS_{\beta_{s_y}}^k}^{\text{unknown}} + \overbrace{K_{u_z s_z}^{k\alpha_{u_z}\beta_{s_z}}}^{1\times1\,\text{kernel}} \overbrace{{}^zS_{\beta_{s_z}}^k}^{\text{unknown}} = \overbrace{{}^zR_{\alpha_{u_z}}^k}^{\text{known}} \\
\overbrace{K_{s_x u_x}^{k\alpha_{s_x}\beta_{u_x}}}^{1\times1\,\text{kernel}} \overbrace{{}^xU_{\beta_{u_x}}^k}^{\text{unknown}} + \overbrace{K_{s_x u_z}^{k\alpha_{s_x}\beta_{u_z}}}^{1\times1\,\text{kernel}} \overbrace{{}^zU_{\beta_{u_z}}^k}^{\text{unknown}} + \overbrace{K_{s_x s_x}^{k\alpha_{s_x}\beta_{s_x}}}^{1\times1\,\text{kernel}} \overbrace{{}^xS_{\beta_{s_x}}^k}^{\text{unknown}} = 0 \\
\overbrace{K_{s_y u_y}^{k\alpha_{s_y}\beta_{u_y}}}^{1\times1\,\text{kernel}} \overbrace{{}^yU_{\beta_{u_y}}^k}^{\text{unknown}} + \overbrace{K_{s_y u_z}^{k\alpha_{s_y}\beta_{u_z}}}^{1\times1\,\text{kernel}} \overbrace{{}^zU_{\beta_{u_z}}^k}^{\text{unknown}} + \overbrace{K_{s_y s_y}^{k\alpha_{s_y}\beta_{s_y}}}^{1\times1\,\text{kernel}} \overbrace{{}^yS_{\beta_{s_y}}^k}^{\text{unknown}} = 0 \\
\overbrace{K_{s_z u_x}^{k\alpha_{s_z}\beta_{u_x}}}^{1\times1\,\text{kernel}} \overbrace{{}^xU_{\beta_{u_x}}^k}^{\text{unknown}} + \overbrace{K_{s_z u_y}^{k\alpha_{s_z}\beta_{u_y}}}^{1\times1\,\text{kernel}} \overbrace{{}^yU_{\beta_{u_y}}^k}^{\text{unknown}} + \overbrace{K_{s_z u_z}^{k\alpha_{s_z}\beta_{u_z}}}^{1\times1\,\text{kernel}} \overbrace{{}^zU_{\beta_{u_z}}^k}^{\text{unknown}} + \overbrace{K_{s_z s_z}^{k\alpha_{s_z}\beta_{s_z}}}^{1\times1\,\text{kernel}} \overbrace{{}^zS_{\beta_{s_z}}^k}^{\text{unknown}} = 0
\end{array} \tag{14}$$

It will be shown that Eq. (14) is valid for the case of layerwise theories and also for higher order shear deformation theories (see *Part*

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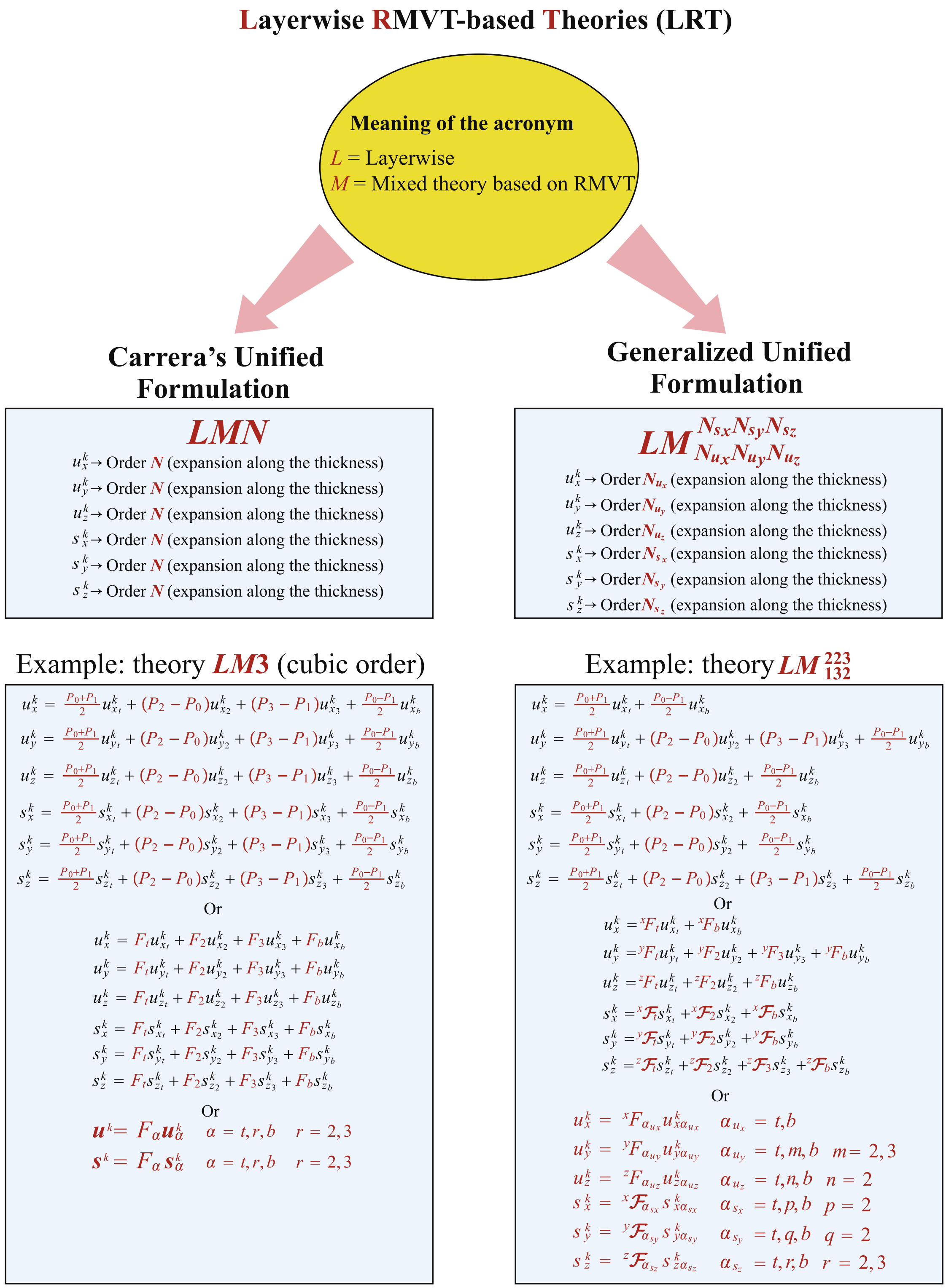

Fig. 2. Acronyms used to define the RMVT-based layerwise theories using Carrera's unified formulation and generalized unified formulation.

III, Ref. [11]) with or without Zig-Zag functions (see *Part IV*, Ref. [12]). The *loads* have been *defined* as follows:

$$
\begin{aligned}
{}^xR^k_{\alpha_{u_x}} &= \overbrace{D^{kt\alpha_{u_x}\beta_{u_x}}_{u_xu_x}}^{1\times1\text{pressure kernel}} \overbrace{{}^xP^{kt}_{\beta_{u_x}}}^{\text{input}} + \overbrace{D^{kb\alpha_{u_x}\beta_{u_x}}_{u_xu_x}}^{1\times1\text{pressure kernel}} \overbrace{{}^xP^{kb}_{\beta_{u_x}}}^{\text{input}} \\
{}^yR^k_{\alpha_{u_y}} &= \overbrace{D^{kt\alpha_{u_y}\beta_{u_y}}_{u_yu_y}}^{1\times1\text{pressure kernel}} \overbrace{{}^yP^{kt}_{\beta_{u_y}}}^{\text{input}} + \overbrace{D^{kb\alpha_{u_y}\beta_{u_y}}_{u_yu_y}}^{1\times1\text{pressure kernel}} \overbrace{{}^yP^{kb}_{\beta_{u_y}}}^{\text{input}} \\
{}^zR^k_{\alpha_{u_z}} &= \overbrace{D^{kt\alpha_{u_z}\beta_{u_z}}_{u_zu_z}}^{1\times1\text{pressure kernel}} \overbrace{{}^zP^{kt}_{\beta_{u_z}}}^{\text{input}} + \overbrace{D^{kb\alpha_{u_z}\beta_{u_z}}_{u_zu_z}}^{1\times1\text{pressure kernel}} \overbrace{{}^zP^{kb}_{\beta_{u_z}}}^{\text{input}}
\end{aligned}
\tag{15}
$$

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The amplitudes of the type ${}^{x}P^{kt}_{\beta_{u_x}}$ are assigned by the user. This input is given at multilayer level, as will be shown.

Eqs. 14 and 15 are *invariant* with respect to the theory. That is, theories LM^{546}_{324} and LM^{626}_{334} (among the infinite possible theories) are generated from Eqs. 14 and 15. Where is the difference between the two theories? The difference is at *layer level*, after the kernels have been expanded to have the layer matrices. These matrices then have to be assembled at multilayer level.

3.2. Expansion of the 1 × 1 kernels: matrices at layer level

First, it has to be pointed out that so far each layer is treated using the same functions. Therefore, the number of terms used to describe the layer matrix is kept the same. This is not mandatory, but unless local effects have to be taken into account (e.g., delamination) it is a "natural" choice. Even in the case of local effects a sufficiently high order can be used and the usage of different orders for the different layers can be avoided. In this paper the expansion used in the different variables does not change and each layer is treated in the same way. Thus, for example, $N^k_{u_x} = N^{k+1}_{u_x} = N_{u_x}$. The expansion of the kernels is the most important part of the generation of one of the possible ∞^6 theories. This operation is done *at layer level*. To explain how this operation is performed, consider the case of theory LM^{546}_{324}, in which the number of degrees of freedom, *at layer level*, is the following:

$$[\mathrm{DOF}]^k_{u_x} = N_{u_x} + 1 = 3 + 1 = 4 \qquad [\mathrm{DOF}]^k_{u_y} = N_{u_y} + 1 = 2 + 1 = 3$$
$$[\mathrm{DOF}]^k_{u_z} = N_{u_z} + 1 = 4 + 1 = 5 \qquad [\mathrm{DOF}]^k_{s_x} = N_{s_x} + 1 = 5 + 1 = 6$$
$$[\mathrm{DOF}]^k_{s_y} = N_{s_y} + 1 = 4 + 1 = 5 \qquad [\mathrm{DOF}]^k_{s_z} = N_{s_z} + 1 = 6 + 1 = 7 \tag{16}$$

From the number of degrees of freedom it is possible to calculate the size of the layer matrices. For example, when matrix $K^{k\alpha_{u_x}\beta_{s_z}}_{u_x s_z}$ is expanded then the final size at layer level will be $[\mathrm{DOF}]^k_{u_x} \times [\mathrm{DOF}]^k_{s_z}$. In the example relative to theory LM^{546}_{324}, matrix $K^{k\alpha_{u_x}\beta_{s_z}}_{u_x s_z}$ at layer level (indicated as $\mathbf{K}^k_{u_x s_z}$) is a 4×7 matrix and obtained as explained in Fig. 3.

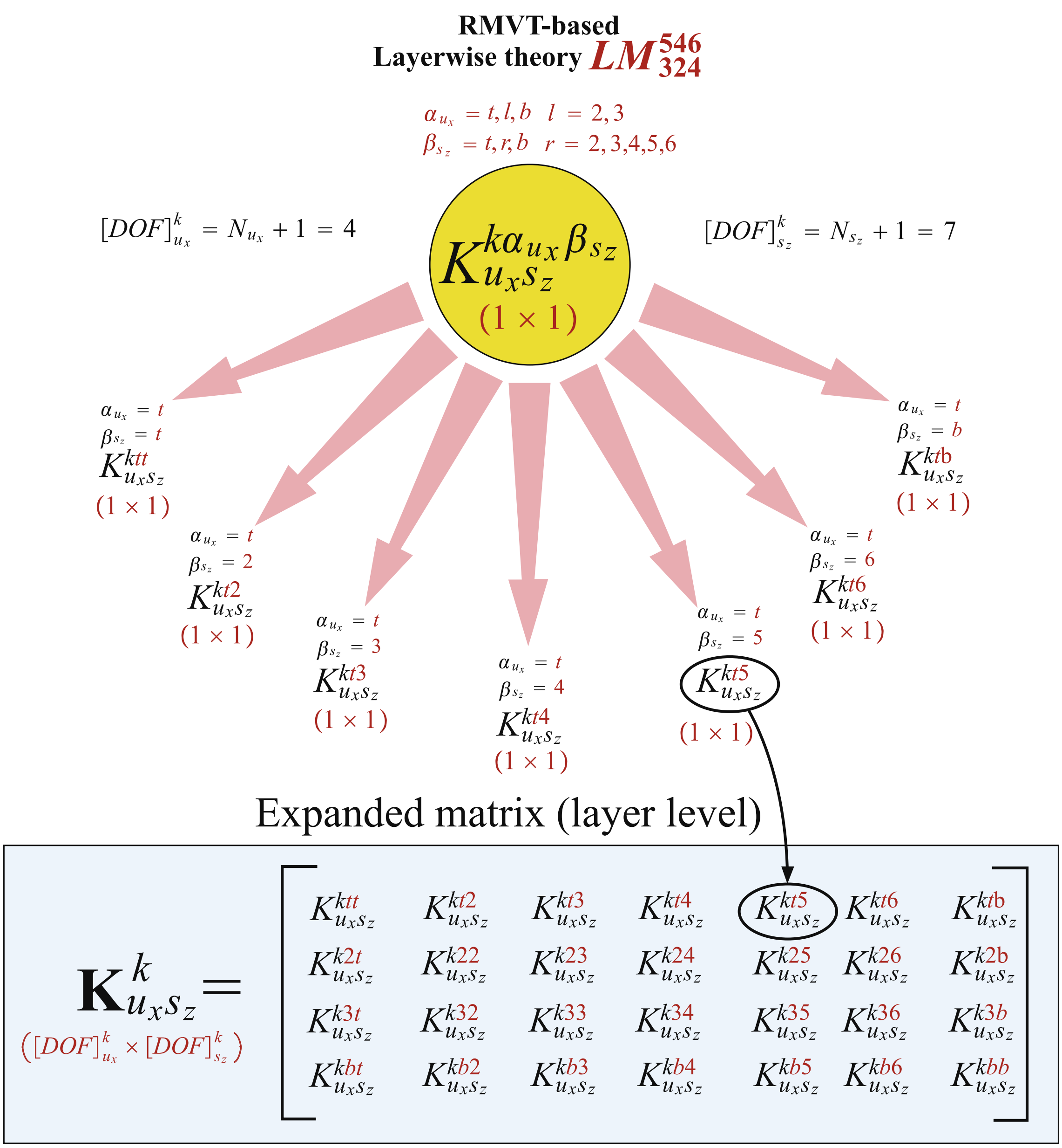

Fig. 3. Generalized unified formulation: example of expansion from a kernel to a layer matrix. Case of theory LM^{546}_{324}. From $K^{k\alpha_{u_x}\beta_{s_z}}_{u_x s_z}$ to $\mathbf{K}^k_{u_x s_z}$.

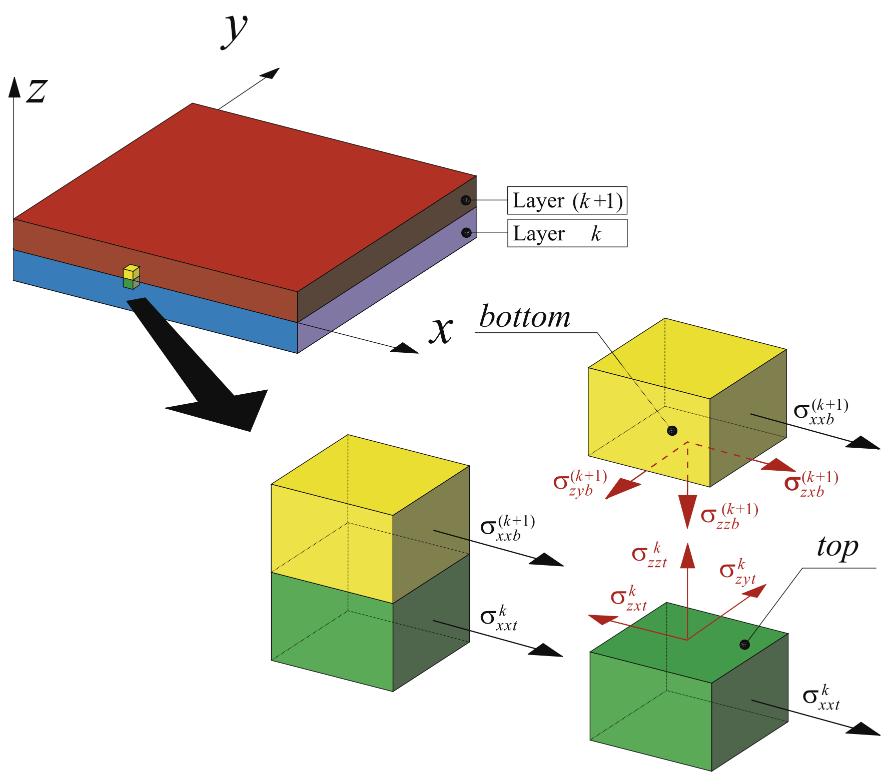

Fig. 4. Equilibrium between two adjacent layers.

3.3. Assembling in the thickness direction: from layer to multilayer level

In addition to the compatibility of the displacements, the equilibrium between two adjacent layers implies that $s^k_{x_t} = s^{(k+1)}_{x_b}$, $s^k_{y_t} = s^{(k+1)}_{y_b}$ and $s^k_{z_t} = s^{(k+1)}_{z_b}$ (see Fig. 4). Therefore, the assembling must consider this fact. Fig. 5 shows how the assembling of a typical matrix is performed. The pressure matrices are obtained from the pressure kernels using the same method shown in Figs. 3 and 5. The use of combinations of Legendre polynomials ensures the continuity of the functions, which are used to expand the displacements and stresses. Fig. 6 shows this concept for the case of theory LM^{546}_{324}. The pressure amplitudes at multilayer level are *input* of the problem. Some input examples are shown in Fig. 7.

Once the matrices are all assembled, the system of equations becomes:

$$\begin{bmatrix} \mathbf{K}_{u_xu_x} & \mathbf{K}_{u_xu_y} & \mathbf{0}_{u_xu_z} & \mathbf{K}_{u_xs_x} & \mathbf{0}_{u_xs_y} & \mathbf{K}_{u_xs_z} \\ & \mathbf{K}_{u_yu_y} & \mathbf{0}_{u_yu_z} & \mathbf{0}_{u_ys_x} & \mathbf{K}_{u_ys_y} & \mathbf{K}_{u_ys_z} \\ & & \mathbf{0}_{u_zu_z} & \mathbf{K}_{u_zs_x} & \mathbf{K}_{u_zs_y} & \mathbf{K}_{u_zs_z} \\ & & & \mathbf{K}_{s_xs_x} & \mathbf{0}_{s_xs_y} & \mathbf{0}_{s_xs_z} \\ \text{Symm} & & & & \mathbf{K}_{s_ys_y} & \mathbf{0}_{s_ys_z} \\ & & & & & \mathbf{K}_{s_zs_z} \end{bmatrix} \cdot \begin{bmatrix} {}^x\mathbf{U} \\ {}^y\mathbf{U} \\ {}^z\mathbf{U} \\ {}^x\mathbf{S} \\ {}^y\mathbf{S} \\ {}^z\mathbf{S} \end{bmatrix} = \begin{bmatrix} {}^x\mathbf{R} \\ {}^y\mathbf{R} \\ {}^z\mathbf{R} \\ {}^x\mathbf{0} \\ {}^y\mathbf{0} \\ {}^z\mathbf{0} \end{bmatrix} \tag{17}$$

where

$$\begin{aligned} {}^x\mathbf{R} &= {}^t\mathbf{D}_{u_xu_x} \cdot {}^x\mathbf{P}^t + {}^b\mathbf{D}_{u_xu_x} \cdot {}^x\mathbf{P}^b \\ {}^y\mathbf{R} &= {}^t\mathbf{D}_{u_yu_y} \cdot {}^y\mathbf{P}^t + {}^b\mathbf{D}_{u_yu_y} \cdot {}^y\mathbf{P}^b \\ {}^z\mathbf{R} &= {}^t\mathbf{D}_{u_zu_z} \cdot {}^z\mathbf{P}^t + {}^b\mathbf{D}_{u_zu_z} \cdot {}^z\mathbf{P}^b \end{aligned} \tag{18}$$

4. Theoretical examples

The generation of the infinite theories is not a very difficult problem when the generalized unified formulation is used. To help the readers create their own code based on this procedure, the author reports here several numerical examples. In particular, theory LM^{546}_{324} will be analyzed in detail. Suppose that the goal is the generation of matrix $\mathbf{K}_{u_xs_z}$.

The *kernel* associated with matrix $\mathbf{K}_{u_xs_z}$ (at layer level) is the following:

$$K^{k\alpha_{u_x}\beta_{s_z}}_{u_xs_z} = -\frac{m\pi}{a} Z^{k\alpha_{u_x}\beta_{s_z}}_{13u_xs_z} = -\frac{m\pi}{a} C^k_{13} \int_{z_{\text{bot}_k}}^{z_{\text{top}_k}} {}^xF_{\alpha_{u_x}}(z)\, {}^z\mathcal{F}_{\alpha_{s_z}}(z)\mathrm{d}z \tag{19}$$

The functions used in the expansions along the thickness are defined as a combination of Legendre polynomials. Therefore, it is convenient to transform the interval (see Eq. 8). The kernel is then

$$K^{k\alpha_{u_x}\beta_{s_z}}_{u_xs_z} = -\frac{m\pi}{a} C^k_{13} \frac{h_k}{2} \int_{-1}^{+1} {}^xF_{\alpha_{u_x}}(\zeta_k)\, {}^z\mathcal{F}_{\alpha_{s_z}}(\zeta_k)\mathrm{d}\zeta_k \tag{20}$$

For the case in which $\alpha_{u_x} = \beta_{s_z} = t$:

$$\begin{aligned} K^{ktt}_{u_xs_z} &= -\tfrac{m\pi}{a} C^k_{13} \tfrac{h_k}{2} \textstyle\int_{-1}^{+1} {}^xF_t(\zeta_k)\, {}^z\mathcal{F}_t(\zeta_k)\mathrm{d}\zeta_k \\ &= -\tfrac{m\pi}{a} C^k_{13} \tfrac{h_k}{2} \textstyle\int_{-1}^{+1} \tfrac{P_0+P_1}{2} \cdot \tfrac{P_0+P_1}{2} \mathrm{d}\zeta_k \end{aligned} \tag{21}$$

or

$$K^{ktt}_{u_xs_z} = -\frac{m\pi}{a} C^k_{13} \frac{h_k}{2} \int_{-1}^{+1} \left(\frac{1+\zeta_k}{2}\right)^2 \mathrm{d}\zeta_k = -C^k_{13} \frac{m\pi h_k}{3a} \tag{22}$$

Similarly for the case in which $\alpha_{u_x} = t$ and $\beta_{s_z} = 2$:

$$\begin{aligned} K^{kt2}_{u_xs_z} &= -\tfrac{m\pi}{a} C^k_{13} \tfrac{h_k}{2} \textstyle\int_{-1}^{+1} {}^xF_t(\zeta_k)\, {}^z\mathcal{F}_2(\zeta_k)\mathrm{d}\zeta_k \\ &= -\tfrac{m\pi}{a} C^k_{13} \tfrac{h_k}{2} \textstyle\int_{-1}^{+1} \tfrac{P_0+P_1}{2} \cdot (P_2 - P_0)\mathrm{d}\zeta_k \end{aligned} \tag{23}$$

or

$$K^{kt2}_{u_xs_z} = -\frac{m\pi}{a} C^k_{13} \frac{h_k}{2} \int_{-1}^{+1} \frac{1+\zeta_k}{2} \cdot \frac{3(\zeta_k)^2 - 3}{2} \mathrm{d}\zeta_k = \frac{m\pi}{a} C^k_{13} \frac{h_k}{2} \tag{24}$$

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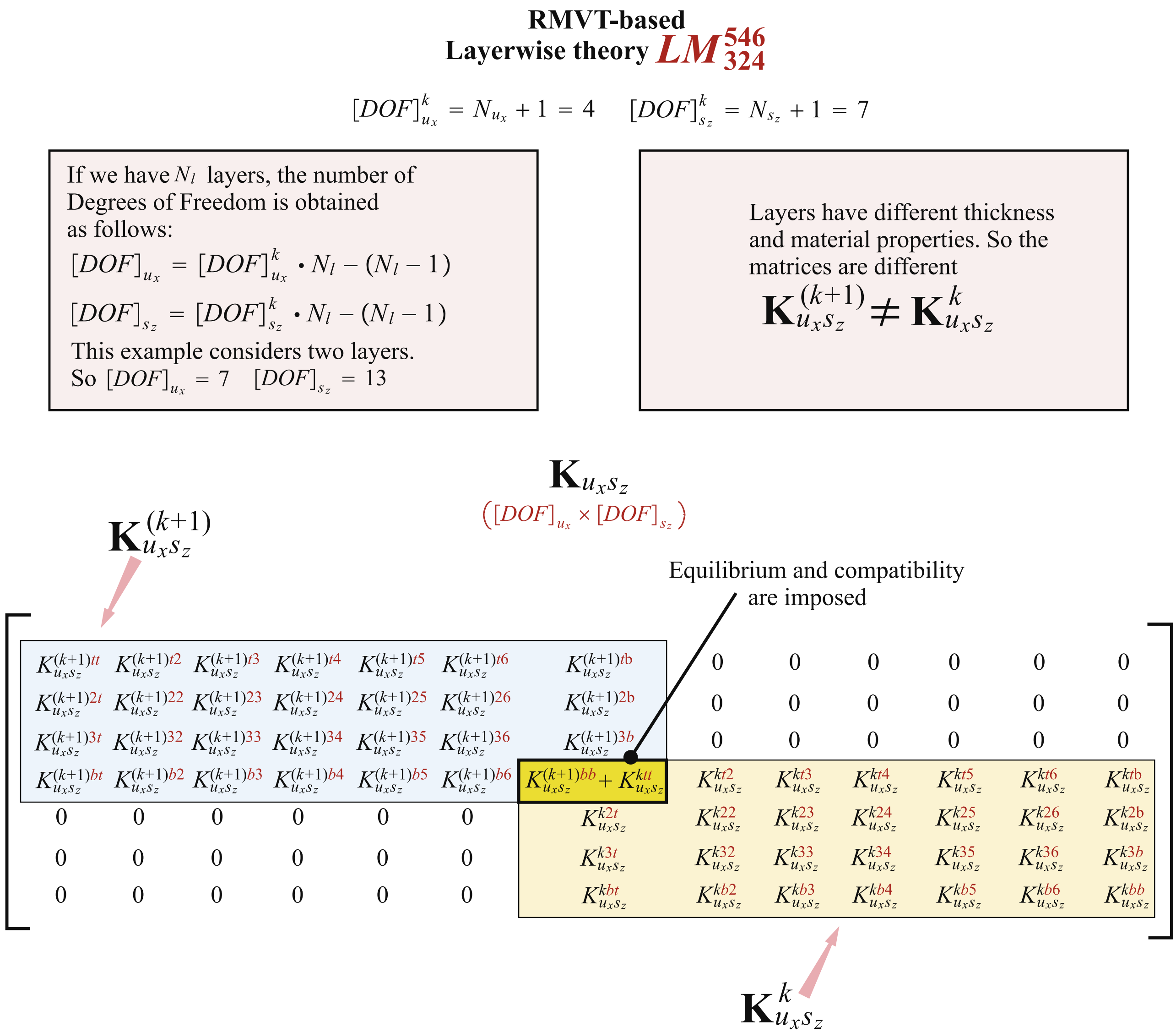

Fig. 5. Generalized unified formulation: example of assembling from layer matrices to multilayer matrix. Case of theory LM_{324}^{546}. From $\mathbf{K}_{u_x s_z}^{k}$ and $\mathbf{K}_{u_x s_z}^{(k+1)}$ to $\mathbf{K}_{u_x s_z}$.

Thirteen independent matrices are required to solve the problem. $\mathbf{K}_{u_x s_z}^{k}$ is just one of them. The other matrices can be calculated with a similar procedure. Another case is analyzed to clarify the procedure. Consider the kernel used to generate $\mathbf{K}_{u_x s_x}^{k}$:

$$K_{u_x s_x}^{k\alpha_{u_x}\beta_{s_x}} = +Z_{u_x s_x}^{k\alpha_{u_x z}\beta_{s_x}} = \int_{z_{\text{bot}_k}}^{z_{\text{top}_k}} \frac{\mathrm{d}[{}^xF_{\alpha_{u_x}}(z_k)]}{\mathrm{d}z_k}\, {}^z\mathcal{F}_{\alpha_{s_x}}(z_k)\mathrm{d}z_k \tag{25}$$

changing the variables and using the thickness coordinate ζ_k:

$$K_{u_x s_x}^{k\alpha_{u_x}\beta_{s_x}} = \int_{-1}^{+1} \frac{\mathrm{d}[{}^xF_{\alpha_{u_x}}(\zeta_k)]}{\mathrm{d}\zeta_k}\, {}^z\mathcal{F}_{\alpha_{s_x}}(\zeta_k)\mathrm{d}\zeta_k \tag{26}$$

The other calculations are omitted for brevity. The 13 independent matrices at layer level are reported in Appendix A.

The sizes of the layer pressure matrix ${}^t\mathbf{D}_{u_x u_x}^k$ and ${}^b\mathbf{D}_{u_x u_x}^k$ are the same as the size of matrix $\mathbf{K}_{u_x u_x}^k$. Similarly, the sizes of the matrices ${}^t\mathbf{D}_{u_y u_y}^k$ and ${}^b\mathbf{D}_{u_y u_y}^k$ are the same as the size of matrix $\mathbf{K}_{u_y u_y}^k$. Finally, the sizes of ${}^t\mathbf{D}_{u_z u_z}^k$ and ${}^b\mathbf{D}_{u_z u_z}^k$ are the same as the size of matrix $\mathbf{K}_{u_z u_z}^k$. The pressures can be applied only at the top or bottom surfaces of the plate. This means that the pressure matrices at layer level are calculated only for $k = N_l$ and $k = 1$, the top and bottom layers respectively. In particular, ${}^t\mathbf{D}_{u_x u_x}^k$, ${}^t\mathbf{D}_{u_y u_y}^k$ and ${}^t\mathbf{D}_{u_z u_z}^k$ are calculated only for $k = N_l$ (for the other layers these matrices are set to be with only zeros). Similarly, ${}^b\mathbf{D}_{u_x u_x}^k$, ${}^b\mathbf{D}_{u_y u_y}^k$ and ${}^b\mathbf{D}_{u_z u_z}^k$ are calculated only for $k = 1$ (for the other layers these matrices are set to be with only zeros). The assembling to multilayer level is then done as for the corresponding matrices. For example, matrix ${}^t\mathbf{D}_{u_x u_x}$ is built using the same procedure used for matrix $\mathbf{K}_{u_x u_x}^k$.

The pressure matrices at layer level are obtained using the definitions reported in *Part I* Ref. [16]. For example, for the top layer $k = N_l$ (notice that in any case the top surface of the layer is found when $\zeta_k = +1$):

$${}^tD_{u_x u_x}^{k=N_l\ \alpha_{u_x}\beta_{u_x}} = {}^xF_{\alpha_{u_x}}^t\, {}^xF_{\beta_{u_x}}^t = {}^xF_{\alpha_{u_x}}(\zeta_k = +1)\, {}^xF_{\beta_{u_x}}(\zeta_k = +1) \tag{27}$$

Considering the properties of the functions along the thickness (combination of Legendre polynomials), the only term that is different than zero is the one corresponding to $\alpha_{u_x} = \beta_{u_x} = t$:

$${}^tD_{u_x u_x}^{k=N_l\ tt} = {}^xF_t^t\, {}^xF_t^t = {}^xF_t(\zeta_k = +1)\, {}^xF_t(\zeta_k = +1) = 1 \tag{28}$$

For the other pressure matrices of the top layer:

$$\begin{aligned} {}^tD_{u_y u_y}^{k=N_l\ tt} &= {}^yF_t^t\, {}^yF_t^t = {}^yF_t(\zeta_k = +1)\, {}^yF_t(\zeta_k = +1) = 1 \\ {}^tD_{u_z u_z}^{k=N_l\ tt} &= {}^zF_t^t\, {}^zF_t^t = {}^zF_t(\zeta_k = +1)\, {}^zF_t(\zeta_k = +1) = 1 \end{aligned} \tag{29}$$

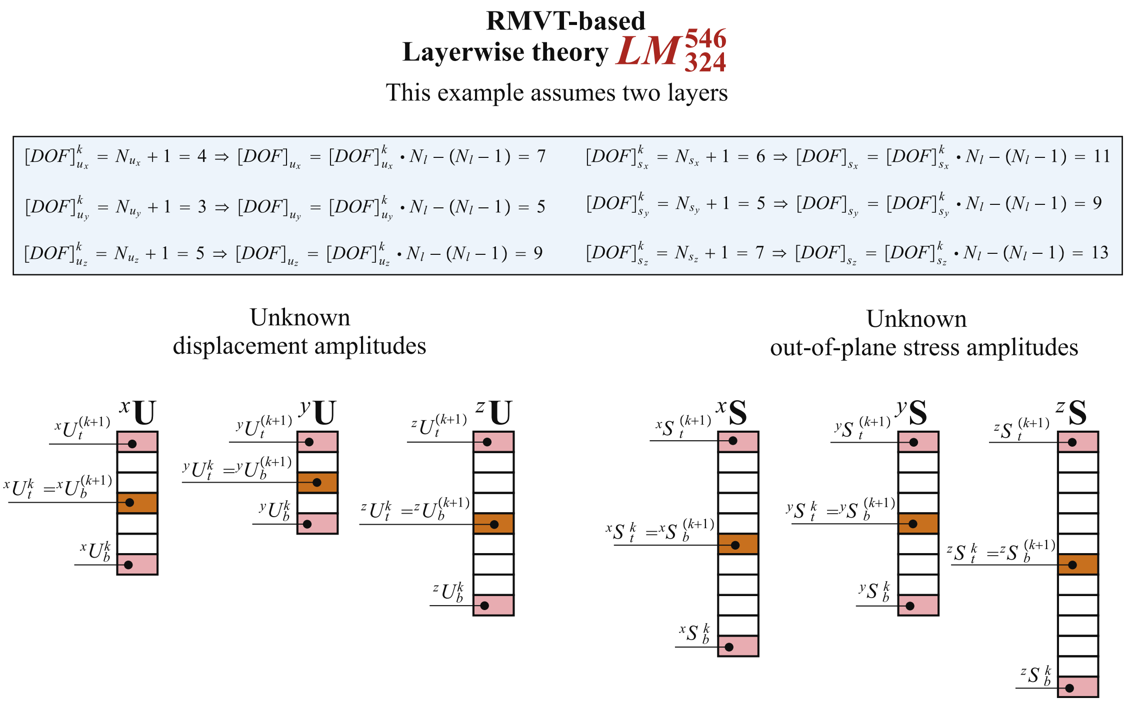

Fig. 6. Case of theory LM_{324}^{546}. Multilayer unknown displacement and out-of-plane stresses for the case in which the number of layers is two.

Considering again the properties of the functions used for the thickness expansions it is deduced that for the bottom layer ($k = 1$) the only terms that are different than zero are the ones with master indices equal to b:

$$\begin{aligned}
{}^bD_{u_xu_x}^{k=1\ bb} &= {}^xF_b^b\ {}^xF_b^b = {}^xF_b(\zeta_k = -1)\ {}^xF_b(\zeta_k = -1) = 1 \\
{}^bD_{u_yu_y}^{k=1\ bb} &= {}^yF_b^b\ {}^yF_b^b = {}^yF_b(\zeta_k = -1)\ {}^yF_b(\zeta_k = -1) = 1 \\
{}^bD_{u_zu_z}^{k=1\ bb} &= {}^zF_b^b\ {}^zF_b^b = {}^zF_b(\zeta_k = -1)\ {}^zF_b(\zeta_k = -1) = 1
\end{aligned} \tag{30}$$

5. Calculation of the stresses

The system of equations with unknown amplitudes of displacements and stresses is represented by Eq. (17). This system can either be directly solved or the static condensation technique (see *part I*, Ref. [16]) can be performed. When the Navier-type solution is considered it is not really important if the static condensation is performed or not, but when FEM computations are concerned, the static condensation (if performed at element level) can significantly save CPU time. Suppose now the amplitudes are known. The stresses need to be calculated. Once the amplitudes are known, it is possible to extract the vectors of amplitudes at *layer level* (see Fig. 6). Form these vectors the displacements and out-of-plane stresses can be calculated immediately using the definition of generalized unified formulation. For the generic layer k the following formulas are valid:

$$\begin{aligned}
u_x^k &= {}^xF_{\alpha_{u_x}} u_{x\alpha_{u_x}}^k = {}^xF_{\alpha_{u_x}} {}^xU_{\alpha_{u_x}}^k \mathcal{C}_a^{m\pi x} \mathcal{S}_b^{n\pi y} \\
u_y^k &= {}^yF_{\alpha_{u_y}} u_{y\alpha_{u_y}}^k = {}^yF_{\alpha_{u_y}} {}^yU_{\alpha_{u_y}}^k \mathcal{S}_a^{m\pi x} \mathcal{C}_b^{n\pi y} \\
u_z^k &= {}^zF_{\alpha_{u_z}} u_{z\alpha_{u_z}}^k = {}^zF_{\alpha_{u_z}} {}^zU_{\alpha_{u_z}}^k \mathcal{S}_a^{m\pi x} \mathcal{S}_b^{n\pi y} \\
s_x^k &= {}^x\mathcal{F}_{\alpha_{s_x}} s_{x\alpha_{s_x}}^k = {}^x\mathcal{F}_{\alpha_{s_x}} {}^xS_{\alpha_{s_x}}^k \mathcal{C}_a^{m\pi x} \mathcal{S}_b^{n\pi y} \\
s_y^k &= {}^y\mathcal{F}_{\alpha_{s_y}} s_{y\alpha_{s_y}}^k = {}^y\mathcal{F}_{\alpha_{s_y}} {}^yS_{\alpha_{s_y}}^k \mathcal{S}_a^{m\pi x} \mathcal{C}_b^{n\pi y} \\
s_z^k &= {}^z\mathcal{F}_{\alpha_{s_z}} s_{z\alpha_{s_z}}^k = {}^z\mathcal{F}_{\alpha_{s_z}} {}^zS_{\alpha_{s_z}}^k \mathcal{S}_a^{m\pi x} \mathcal{S}_b^{n\pi y}
\end{aligned} \tag{31}$$

where, for example, $\mathcal{C}_a^{m\pi x} = \cos\frac{m\pi x}{a}$ and $\mathcal{S}_a^{m\pi x} = \sin\frac{m\pi x}{a}$.

5.1. In-plane stresses calculated using mixed form of Hooke's Law (MFHL)

Since a mixed approach has been adopted, it is a natural choice to use the MFHL (which is explicitly shown in *Part I*, Ref. [16]) to calculate the in-plane stresses:

$$\begin{aligned}
\sigma_{xx}^k &= \Big[-\tfrac{m\pi}{a} C_{11}^k {}^xF_{\alpha_{u_x}} {}^xU_{\alpha_{u_x}}^k - \tfrac{n\pi}{b} C_{12}^k {}^yF_{\alpha_{u_y}} {}^yU_{\alpha_{u_y}}^k \\
&\quad + C_{13}^k {}^z\mathcal{F}_{\alpha_{s_z}} {}^zS_{\alpha_{s_z}}^k\Big] \mathcal{S}_a^{m\pi x} \mathcal{S}_b^{n\pi y} \\
\sigma_{yy}^k &= \Big[-\tfrac{m\pi}{a} C_{12}^k {}^xF_{\alpha_{u_x}} {}^xU_{\alpha_{u_x}}^k - \tfrac{n\pi}{b} C_{22}^k {}^yF_{\alpha_{u_y}} {}^yU_{\alpha_{u_y}}^k \\
&\quad + C_{23}^k {}^z\mathcal{F}_{\alpha_{s_z}} {}^zS_{\alpha_{s_z}}^k\Big] \mathcal{S}_a^{m\pi x} \mathcal{S}_b^{n\pi y} \\
\sigma_{xy}^k &= \Big[+\tfrac{n\pi}{b} C_{66}^k {}^xF_{\alpha_{u_x}} {}^xU_{\alpha_{u_x}}^k + \tfrac{m\pi}{a} C_{66}^k {}^yF_{\alpha_{u_y}} {}^yU_{\alpha_{u_y}}^k\Big] \mathcal{C}_a^{m\pi x} \mathcal{C}_b^{n\pi y}
\end{aligned} \tag{32}$$

The following can be observed:

- The stresses σ_{xx}^k and σ_{yy}^k do not have *explicit* dependence on the amplitudes ${}^zU_{\alpha_{u_z}}^k$, ${}^xS_{\alpha_{s_x}}^k$ and ${}^yS_{\alpha_{s_y}}^k$.
- The stress σ_{xy}^k does not have *explicit* dependence on the amplitudes ${}^zU_{\alpha_{u_z}}^k$, ${}^xS_{\alpha_{s_x}}^k$, ${}^yS_{\alpha_{s_y}}^k$ and ${}^zS_{\alpha_{s_z}}^k$.

Even if there is no explicit dependence on some amplitudes, this fact does not mean that the orders used for the corresponding variables do not affect the result. In fact, the orders used for the other variables change the solution of Eq. (17) which, therefore, affects *all* the quantities.

5.2. In-plane stresses calculated using classical form of Hooke's law (CFHL)

Even if the formulation is based on a mixed approach, it is possible to use CFHL (see Part I, Ref. [16]) to calculate the in-plane stresses. If this approach is chosen, the stresses can be calculated as

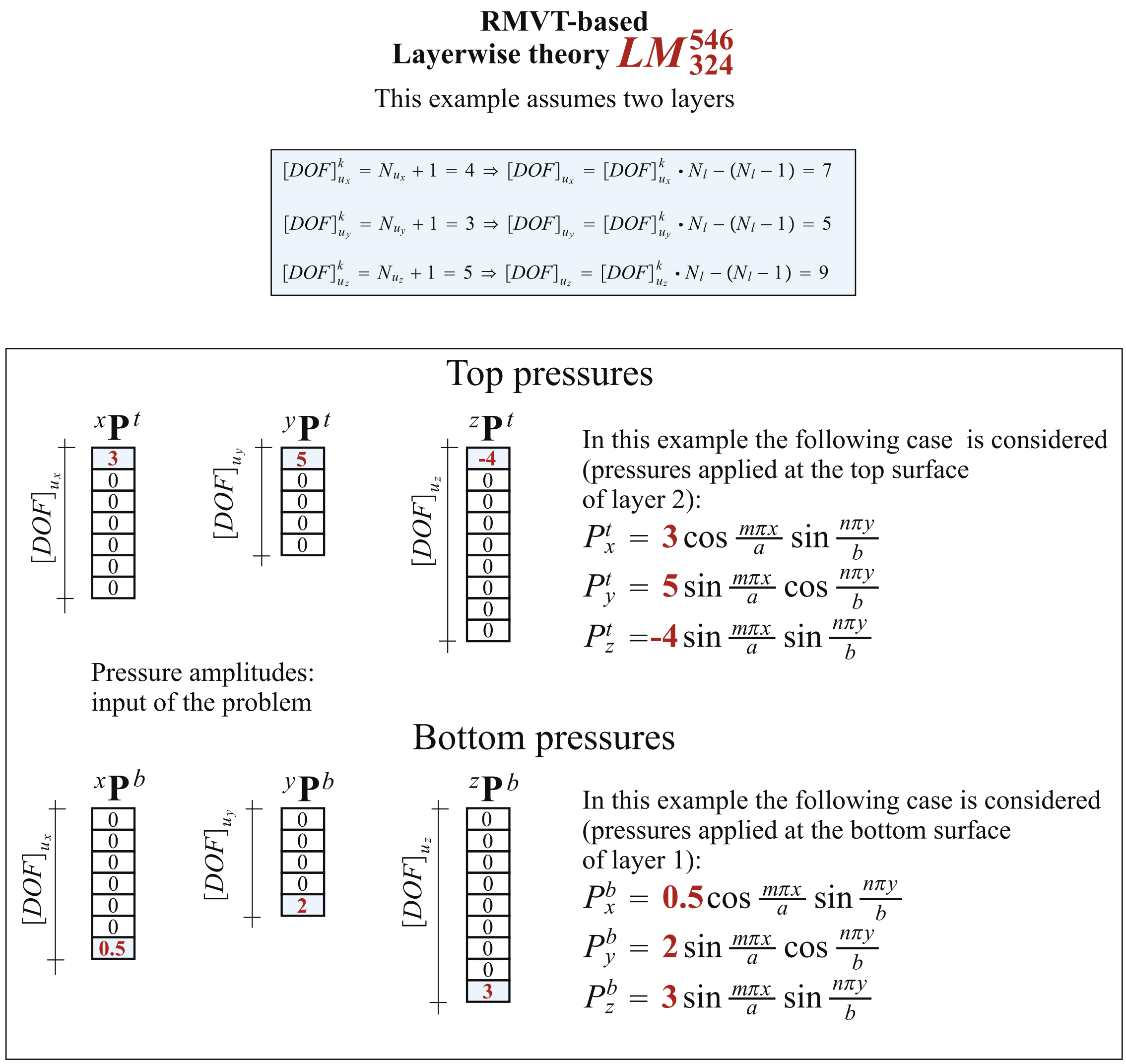

Fig. 7. Case of theory LM_{324}^{546}. Example of pressure amplitudes and inputs at multilayer level for the case in which the number of layers is two.

$$
\begin{aligned}
\sigma^k_{xx} &= \Big[-\tfrac{m\pi}{a}\widetilde{C}^k_{11}\,{}^xF_{\alpha_{ux}}\,{}^xU^k_{\alpha_{ux}} - \tfrac{n\pi}{b}\widetilde{C}^k_{12}\,{}^yF_{\alpha_{uy}}\,{}^yU^k_{\alpha_{uy}} \\
&\quad + \widetilde{C}^k_{13}\,{}^zF_{\alpha_{uz},z}\,{}^zU^k_{\alpha_{uz}}\Big] S^{m\pi x}_a S^{n\pi y}_b \\
\sigma^k_{yy} &= \Big[-\tfrac{m\pi}{a}\widetilde{C}^k_{12}\,{}^xF_{\alpha_{ux}}\,{}^xU^k_{\alpha_{ux}} - \tfrac{n\pi}{b}\widetilde{C}^k_{22}\,{}^yF_{\alpha_{uy}}\,{}^yU^k_{\alpha_{uy}} \\
&\quad + \widetilde{C}^k_{23}\,{}^zF_{\alpha_{uz},z}\,{}^zU^k_{\alpha_{uz}}\Big] S^{m\pi x}_a S^{n\pi y}_b \\
\sigma^k_{xy} &= \Big[+\tfrac{n\pi}{b}\widetilde{C}^k_{66}\,{}^xF_{\alpha_{ux}}\,{}^xU^k_{\alpha_{ux}} + \tfrac{m\pi}{a}\widetilde{C}^k_{66}\,{}^yF_{\alpha_{uy}}\,{}^yU^k_{\alpha_{uy}}\Big] C^{m\pi x}_a C^{n\pi y}_b
\end{aligned}
\tag{33}
$$

The following can be observed:

- The stresses σ^k_{xx} and σ^k_{yy} do not have *explicit* dependence on the amplitudes ${}^xS^k_{\alpha_{sx}}$, ${}^yS^k_{\alpha_{sy}}$ and ${}^zS^k_{\alpha_{sz}}$. This is a direct consequence of the fact that the displacement-based formulas (the CFHL) have been used.
- The stress σ^k_{xy} does not have *explicit* dependence on the amplitudes ${}^zU^k_{\alpha_{uz}}$, ${}^xS^k_{\alpha_{sx}}$, ${}^yS^k_{\alpha_{sy}}$ and ${}^zS^k_{\alpha_{sz}}$. This property was found also when MFHL was used.

Calculating the in-plane stresses by using the classical form of Hooke's law (CFHL) instead of the mixed form of Hooke's Law (MFHL) is in theory not consistent because a mixed approach is used. However, for a "converged" case using either MFHL or CFHL is practically equivalent.

5.3. Out-of-plane stresses calculated using classical form of Hooke's law (CFHL)

The stresses σ^k_{xz}, σ^k_{yz}, σ^k_{zz} can be calculated *a priori* by using Eq. (31) (see in particular the last three expressions). However, it is possible to use CFHL and calculate the stresses. If this procedure is chosen then the following expressions are valid:

$$
\begin{aligned}
\sigma^k_{zx} &= \widetilde{C}^k_{55}\left(\tfrac{m\pi}{a}\,{}^zF_{\alpha_{uz}}\,{}^zU_{\alpha_{uz}} + {}^xF_{\alpha_{ux},z}\,{}^xU_{\alpha_{ux}}\right) C^{m\pi x}_a S^{n\pi y}_b \\
\sigma^k_{zy} &= \widetilde{C}^k_{44}\left(\tfrac{n\pi}{b}\,{}^zF_{\alpha_{uz}}\,{}^zU_{\alpha_{uz}} + {}^yF_{\alpha_{uy},z}\,{}^yU_{\alpha_{uy}}\right) S^{m\pi x}_a C^{n\pi y}_b \\
\sigma^k_{zz} &= -\tfrac{m\pi}{a}\widetilde{C}^k_{13}\,{}^xF_{\alpha_{ux}}\,{}^xU_{\alpha_{ux}} S^{m\pi x}_a S^{n\pi y}_b \\
&\quad -\tfrac{n\pi}{b}\widetilde{C}^k_{23}\,{}^yF_{\alpha_{uy}}\,{}^yU_{\alpha_{uy}} S^{m\pi x}_a S^{n\pi y}_b \\
&\quad +\widetilde{C}^k_{33}\,{}^zF_{\alpha_{uz},z}\,{}^zU_{\alpha_{uz}} S^{m\pi x}_a S^{n\pi y}_b
\end{aligned}
\tag{34}
$$

Strictly speaking this approach is not consistent because the mixed approach (RMVT) is used and the out-of-plane stresses are calculated a priori.

5.4. Out-of-plane stresses calculated by integrating the indefinite equilibrium equations

The out-of-plane stresses can be obtained from the indefinite equilibrium equations as follows:

$$\begin{aligned}&\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{yx}}{\partial y}+\frac{\partial\sigma_{zx}}{\partial z}=0\Rightarrow\frac{\partial\sigma_{zx}}{\partial z}=-\left(\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{yx}}{\partial y}\right)\\&\frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{zy}}{\partial z}=0\Rightarrow\frac{\partial\sigma_{zy}}{\partial z}=-\left(\frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}\right)\\&\frac{\partial\sigma_{xz}}{\partial x}+\frac{\partial\sigma_{yz}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}=0\Rightarrow\frac{\partial\sigma_{zz}}{\partial z}=-\left(\frac{\partial\sigma_{xz}}{\partial x}+\frac{\partial\sigma_{yz}}{\partial y}\right)\end{aligned}\tag{35}$$

and integrating along the thickness of the plate. Two different options are available for the calculation of the shear stresses σ_{zx} and σ_{zy}.

- *Option #1*.
  The out-of-plane shear stresses are calculated by integrating the derivatives of the in-plane stresses obtained by using CFHL. See Eqs. (35) and (33).
- *Option #2*.
  The out-of-plane shear stresses are calculated by integrating the derivatives of the in-plane stresses obtained by using MFHL. See Eqs. (35) and (32).

For the calculation of stress σ_{zz} the following two methods can be used:

- *Method #1*.
  σ_{zz} can be obtained by integrating the derivatives of the out-of-plane shear stresses calculated using CFHL.
- *Method #2*.
  σ_{zz} can be obtained by integrating the derivatives of the out-of-plane shear stresses calculated a priori using the stresses amplitudes.

As for the integration of the indefinite equilibrium equations, this work will use Option #1 for the out-of-plane shear stresses and Method #1 for the stress σ_{zz}. In this case, then, the stresses σ_{zx}, σ_{zy} and σ_{zz} depend explicitly only on the displacement amplitudes.

6. Conclusion

For the first time in the literature, the extension of the generalized unified formulation to the cases of mixed variational statements (in particular Reissner's mixed variational theorem) and layerwise theories is presented. The displacements u_x, u_y, u_z and the stresses σ_{zx}, σ_{zy}, σ_{zz} are expanded along the thickness of each layer by using Legendre polynomials. Each variable can be treated separately from the others. This allows the writing, with a single formal derivation and software, ∞^6 theories. The new methodology allows the user to freely change the orders used for the expansion of the unknowns and to experiment the best combination that better approximates the structural problem under investigation. The proposed approach for layerwise theories is very general and allows to enforce a priori the compatibility of the displacements and the equilibrium between two adjacent layers. These a priori requirements are met by using a particular assembling procedure from layer to multilayer level.

All the theories are generated by expanding 1×1 matrices (the *kernels* of the generalized unified formulation), which are invariant with respect to the theory. Thus, with only 13 matrices (the kernels) ∞^6 theories can be generated without difficulties.

The numerical performances and properties of mixed layerwise theories and generalized unified formulation will be discussed in *Part V* (see [22]) of the present work. In particular, the mixed layerwise theories will be compared against mixed higher order theories and mixed zig-zag theories. Several discussions on numerical stability and the effect of the relative orders used for the stresses and displacements will be discussed. It will be demonstrated that the lessons learned in the layerwise case can be used to interpret the numerical performances of the other types of theories.

Acknowledgement

The author thanks his sister Demasi Paola who inspired him with her strong will.

Appendix A. Expanded matrices at layer level

With the assumption of Navier-type solution and theory LM^{546}_{324}, the 13 independent matrices at layer level can be obtained by expanding the corresponding kernels. The resulting matrices are the following:

$$\mathbf{K}^k_{u_xu_x}=\frac{h_k\pi^2(C^k_{11}b^2m^2+C^k_{66}a^2n^2)}{a^2b^2}\begin{bmatrix}+\frac13&-\frac12&-\frac16&+\frac16\\-\frac12&+\frac65&0&-\frac12\\-\frac16&0&+\frac{10}{21}&+\frac16\\+\frac16&-\frac12&+\frac16&+\frac13\end{bmatrix}\tag{36}$$

$$\mathbf{K}^k_{u_xu_y}=\frac{(C^k_{12}+C^k_{66})h_kmn\pi^2}{ab}\begin{bmatrix}+\frac13&-\frac12&+\frac16\\-\frac12&+\frac65&-\frac12\\-\frac16&0&+\frac16\\+\frac16&-\frac12&+\frac13\end{bmatrix}\tag{37}$$

$$\mathbf{K}^k_{u_xs_x}=\begin{bmatrix}+\frac12&-1&0&0&0&+\frac12\\+1&0&-2&0&0&-1\\0&+2&0&-2&0&0\\-\frac12&+1&0&0&0&-\frac12\end{bmatrix}\tag{38}$$

$$\mathbf{K}^k_{u_xs_z}=\frac{C^k_{13}h_km\pi}{a}\begin{bmatrix}-\frac13&+\frac12&+\frac16&0&0&0&-\frac16\\+\frac12&-\frac65&0&+\frac15&0&0&+\frac12\\+\frac16&0&-\frac{10}{21}&0&+\frac17&0&-\frac16\\-\frac16&+\frac12&-\frac16&0&0&0&-\frac13\end{bmatrix}\tag{39}$$

$$\mathbf{K}^k_{u_yu_y}=\frac{(C^k_{66}b^2m^2+C^k_{22}a^2n^2)h_k\pi^2}{a^2b^2}\begin{bmatrix}+\frac13&-\frac12&+\frac16\\-\frac12&+\frac65&-\frac12\\+\frac16&-\frac12&+\frac13\end{bmatrix}\tag{40}$$

$$\mathbf{K}^k_{u_ys_y}=\begin{bmatrix}+\frac12&-1&0&0&+\frac12\\+1&0&-2&0&-1\\-\frac12&+1&0&0&-\frac12\end{bmatrix}\tag{41}$$

$$\mathbf{K}^k_{u_ys_z}=\frac{C^k_{23}h_kn\pi}{b}\begin{bmatrix}-\frac13&+\frac12&+\frac16&0&0&0&-\frac16\\+\frac12&-\frac65&0&+\frac15&0&0&+\frac12\\-\frac16&+\frac12&-\frac16&0&0&0&-\frac13\end{bmatrix}\tag{42}$$

$$\mathbf{K}^k_{u_zs_x}=\frac{h_km\pi}{a}\begin{bmatrix}+\frac13&-\frac12&-\frac16&0&0&+\frac16\\-\frac12&+\frac65&0&-\frac15&0&-\frac12\\-\frac16&0&+\frac{10}{21}&0&-\frac17&+\frac16\\+0&-\frac15&0&+\frac{14}{45}&0&0\\+\frac16&-\frac12&+\frac16&0&0&+\frac13\end{bmatrix}\tag{43}$$

$$\mathbf{K}^k_{u_zs_y}=\frac{h_kn\pi}{b}\begin{bmatrix}+\frac13&-\frac12&-\frac16&0&+\frac16\\-\frac12&+\frac65&0&-\frac15&-\frac12\\-\frac16&0&+\frac{10}{21}&0&+\frac16\\0&-\frac15&0&+\frac{14}{45}&0\\+\frac16&-\frac12&+\frac16&0&+\frac13\end{bmatrix}\tag{44}$$

$$\mathbf{K}^k_{u_zs_z}=\begin{bmatrix}+\frac12&-1&0&0&0&0&+\frac12\\+1&0&-2&0&0&0&-1\\0&+2&0&-2&0&0&0\\0&0&+2&0&-2&0&0\\-\frac12&+1&0&0&0&0&-\frac12\end{bmatrix}\tag{45}$$

$$\mathbf{K}^k_{s_xs_x}=C^k_{55}h_k\begin{bmatrix}-\frac13&+\frac12&+\frac16&0&0&-\frac16\\+\frac12&-\frac65&0&+\frac15&0&+\frac12\\+\frac16&0&-\frac{10}{21}&0&+\frac17&-\frac16\\0&+\frac15&0&-\frac{14}{45}&0&0\\0&0&+\frac17&0&-\frac{18}{77}&0\\-\frac16&+\frac12&-\frac16&0&0&-\frac13\end{bmatrix}\tag{46}$$

$$\mathbf{K}^k_{s_y s_y} = C^k_{44} h_k \begin{bmatrix} -\frac{1}{3} & +\frac{1}{2} & +\frac{1}{6} & 0 & -\frac{1}{6} \\ +\frac{1}{2} & -\frac{6}{5} & 0 & +\frac{1}{5} & +\frac{1}{2} \\ +\frac{1}{6} & 0 & -\frac{10}{21} & 0 & -\frac{1}{6} \\ 0 & +\frac{1}{5} & 0 & -\frac{14}{45} & 0 \\ -\frac{1}{6} & +\frac{1}{2} & -\frac{1}{6} & 0 & -\frac{1}{3} \end{bmatrix} \tag{47}$$

$$\mathbf{K}^k_{s_z s_z} = C^k_{33} h_k \begin{bmatrix} -\frac{1}{3} & +\frac{1}{2} & +\frac{1}{6} & 0 & 0 & 0 & -\frac{1}{6} \\ +\frac{1}{2} & -\frac{6}{5} & 0 & +\frac{1}{5} & 0 & 0 & +\frac{1}{2} \\ +\frac{1}{6} & 0 & -\frac{10}{21} & 0 & +\frac{1}{7} & 0 & -\frac{1}{6} \\ 0 & +\frac{1}{5} & 0 & -\frac{14}{45} & 0 & +\frac{1}{9} & 0 \\ 0 & 0 & +\frac{1}{7} & 0 & -\frac{18}{77} & 0 & 0 \\ 0 & 0 & 0 & +\frac{1}{9} & 0 & -\frac{22}{117} & 0 \\ -\frac{1}{6} & +\frac{1}{2} & -\frac{1}{6} & 0 & 0 & 0 & -\frac{1}{3} \end{bmatrix} \tag{48}$$

Now consider a numerical example – a plate consists of one layer, with the following properties:

$$m = 2; \quad n = 3; \quad a = 10; \quad b = 15; \quad h = 3; \quad \vartheta = 0 \tag{49}$$

$$\begin{cases} E_{11} = 25 & E_{22} = 4 & E_{33} = 3 \\ G_{12} = \frac{1}{2} & G_{13} = \frac{3}{5} & G_{23} = \frac{1}{5} \\ \upsilon_{12} = \frac{1}{4} & \upsilon_{13} = \frac{27}{100} & \upsilon_{23} = \frac{29}{100} \end{cases} \tag{50}$$

Some of the matrices are numerically calculated and their expressions are reported below:

$$\mathbf{K}_{u_x u_x} = \begin{bmatrix} 10.17 & -15.25 & -5.08 & 5.08 \\ -15.25 & 36.60 & 0 & -15.25 \\ -5.08 & 0 & 14.52 & 5.08 \\ 5.08 & -15.25 & 5.08 & 10.17 \end{bmatrix} \tag{51}$$

$$\mathbf{K}_{u_x u_y} = \begin{bmatrix} 0.60 & -0.89 & 0.30 \\ -0.89 & 2.15 & -0.89 \\ -0.30 & 0 & 0.30 \\ 0.30 & -0.89 & 0.60 \end{bmatrix} \tag{52}$$

$$\mathbf{K}_{u_x s_z} = \begin{bmatrix} -0.22 & 0.33 & 0.11 & 0 & 0 & 0 & -0.11 \\ 0.33 & -0.78 & 0 & 0.13 & 0 & 0 & 0.33 \\ 0.11 & 0 & -0.31 & 0 & 0.09 & 0 & -0.11 \\ -0.11 & 0.33 & -0.11 & 0 & 0 & 0 & -0.22 \end{bmatrix} \tag{53}$$

$$\mathbf{K}_{u_y u_y} = \begin{bmatrix} 1.79 & -2.69 & 0.90 \\ -2.69 & 6.45 & -2.69 \\ 0.90 & -2.69 & 1.79 \end{bmatrix} \tag{54}$$

$$\mathbf{K}_{u_y s_z} = \begin{bmatrix} -0.19 & 0.29 & 0.10 & 0 & 0 & 0 & -0.10 \\ 0.29 & -0.69 & 0 & 0.11 & 0 & 0 & 0.29 \\ -0.10 & 0.29 & -0.10 & 0 & 0 & 0 & -0.19 \end{bmatrix} \tag{55}$$

$$\mathbf{K}_{s_x s_x} = \begin{bmatrix} -1.67 & 2.50 & 0.83 & 0 & 0 & -0.83 \\ 2.50 & -6.00 & 0 & 1.00 & 0 & 2.50 \\ 0.83 & 0 & -2.38 & 0 & 0.71 & -0.83 \\ 0 & 1.00 & 0 & -1.56 & 0 & 0 \\ 0 & 0 & 0.71 & 0 & -1.17 & 0 \\ -0.83 & 2.50 & -0.83 & 0 & 0 & -1.67 \end{bmatrix} \tag{56}$$

$$\mathbf{K}_{s_y s_y} = \begin{bmatrix} -5.00 & 7.50 & 2.50 & 0 & -2.50 \\ 7.50 & -18.00 & 0 & 3.00 & 7.50 \\ 2.50 & 0 & -7.14 & 0 & -2.50 \\ 0 & 3.00 & 0 & -4.67 & 0 \\ -2.50 & 7.50 & -2.5 & 0 & -5.00 \end{bmatrix} \tag{57}$$

$$\mathbf{K}_{s_z s_z} = \begin{bmatrix} -0.31 & 0.46 & 0.15 & 0 & 0 & 0 & -0.15 \\ 0.46 & -1.11 & 0 & 0.18 & 0 & 0 & 0.46 \\ 0.15 & 0 & -0.44 & 0 & 0.13 & 0 & -0.15 \\ 0 & 0.18 & 0 & -0.29 & 0 & 0.10 & 0 \\ 0 & 0 & 0.13 & 0 & -0.22 & 0 & 0 \\ 0 & 0 & 0 & 0.10 & 0 & -0.17 & 0 \\ -0.15 & 0.46 & -0.15 & 0 & 0 & 0 & -0.31 \end{bmatrix} \tag{58}$$

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