Composite Structures xxx (2008) xxx–xxx

Contents lists available at ScienceDirect

Composite Structures

journal homepage: www.elsevier.com/locate/compstruct

∞^6 Mixed plate theories based on the Generalized Unified Formulation. Part III: Advanced mixed high order shear deformation theories

Luciano Demasi *

Department of Aerospace Engineering and Engineering Mechanics, San Diego State University, 5500 Campanile Drive, San Diego, CA 92182, USA

ARTICLE INFO

Keywords:
Generalized unified formulation
Higher order shear deformation plate theories
Mixed formulation
Reissner's Mixed Variational Theorem

ABSTRACT

The Generalized Unified Formulation was introduced in *Part I* in the case of plate theories based upon Reissner's Mixed Variational Theorem. *Part II* analyzed the case of layerwise theories.

In this work (*Part III*) the Generalized Unified Formulation is applied, for the first time in the literature, to the case of mixed higher order shear deformation theories. The displacements u_x, u_y, u_z have an equivalent single layer description, whereas the stresses σ_{zx}, σ_{zy}, σ_{zz} have a layerwise description. The compatibility of the displacements and the equilibrium of the transverse stresses between two adjacent layers are enforced a *priori*. If the out-of-plane stresses are eliminated using the Static Condensation Technique the resulting theories are formally identical to the displacement-based "classical" higher order shear deformation theories. If the static condensation is not applied then a quasi-layerwise higher order theory is obtained. ∞^6 Mixed higher order shear deformation theories are therefore presented. *All* ∞^6 theories are generated by expanding thirteen 1×1 *invariant* matrices (the kernels of the Generalized Unified Formulation). The kernels have the same formal expressions as the ones used for the layerwise theories analyzed in *Part II*. This makes the generation of the present mixed higher order shear deformation theories particularly effective and easy to implement in a computer code.

1. Introduction

1.1. Higher order shear deformation theories: main concepts

Classical theories were originally developed for isotropic thin plates (see [1–3]). The transverse shear and normal strains are considered negligible with respect to the other strains. Classical plate theory (CPT) is based on the following *displacement field*:

$$\begin{aligned} u_x(x,y,z) &= u_{x_0}(x,y) - z\frac{\partial u_{z_0}(x,y)}{\partial x} \\ u_y(x,y,z) &= u_{y_0}(x,y) - z\frac{\partial u_{z_0}(x,y)}{\partial y} \\ u_z(x,y,z) &= u_{z_0}(x,y) \end{aligned} \tag{1}$$

where u_{x_0}, u_{y_0} and u_{z_0} are the displacements of points in the middle plane of the plate. From Eq. (1) it is not difficult to demonstrate that $\gamma_{zx} = \gamma_{zy} = \varepsilon_{zz} = 0$. These classical approaches begin to fail when the plate is considerably thick (tree-dimensional effects) or when local effects are important (for example, with localized loads or change of the geometry). In the case of composite structures (see Fig. 1) there is the additional problem of correctly simulating of the transverse anisotropy and transverse high deformability. Classical models are thus inadequate.

These problems are alleviated if first order shear deformation theories (FSDT) are used (see [4–7]). A possible FSDT has the following displacement field:

$$\begin{aligned} u_x(x,y,z) &= u_{x_0}(x,y) + z\phi_{u_x}(x,y) \\ u_y(x,y,z) &= u_{y_0}(x,y) + z\phi_{u_y}(x,y) \\ u_z(x,y,z) &= u_{z_0}(x,y) \end{aligned} \tag{2}$$

From Eq. (2) it can be immediately deduced that the shear strains γ_{zx} and γ_{zy} do not change in the thickness direction but are *not* zero, as in CPT. However, ε_{zz} is still zero.

The displacement field can be improved if *higher order terms* are added to the expansion of the variables. In such case we have the higher order shear deformation theories (HSDT) (see [8–11]). A possible HSDT is the following:

$$\begin{aligned} u_x(x,y,z) &= u_{x_0}(x,y) + z\phi_{1_{u_x}}(x,y) + z^2\phi_{2_{u_x}}(x,y) \\ u_y(x,y,z) &= u_{y_0}(x,y) + z\phi_{1_{u_y}}(x,y) + z^2\phi_{2_{u_y}}(x,y) \\ u_z(x,y,z) &= u_{z_0}(x,y) \end{aligned} \tag{3}$$

In Eq. (3), γ_{zx} and γ_{zy} are not constant in the thickness direction but ε_{zz} is zero. Many other improvements can be suggested. For example, other terms can be added to the expansion of the out-of-plane displacement u_z; thus, the transverse normal strain is not zero. It is

* Tel.: +1 619 594 3752; fax: +1 619 594 6005.
E-mail address: ldemasi@mail.sdsu.edu

0263-8223/$ - see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.compstruct.2008.07.011

also possible (see [12]) to use *mixed* variational statements and include, as unknowns, some of the stresses. This approach allows to impose the equilibrium of the transverse stresses between two adjacent layers a priori. Other improvements such as zig-zag theories (see review [13]) are discussed in *Part IV* of this work (see [14]).

1.2. What are the new contributions of this work

The Generalized Unified Formulation (GUF) introduced in reference [15] in the case of single-layer isotropic plates was extended to the case of single-layer orthotropic plates in reference [16]. GUF

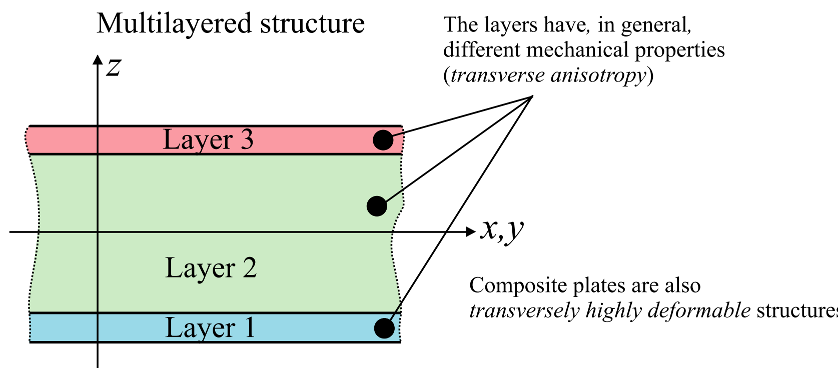

Fig. 1. Multilayered structure: concepts.

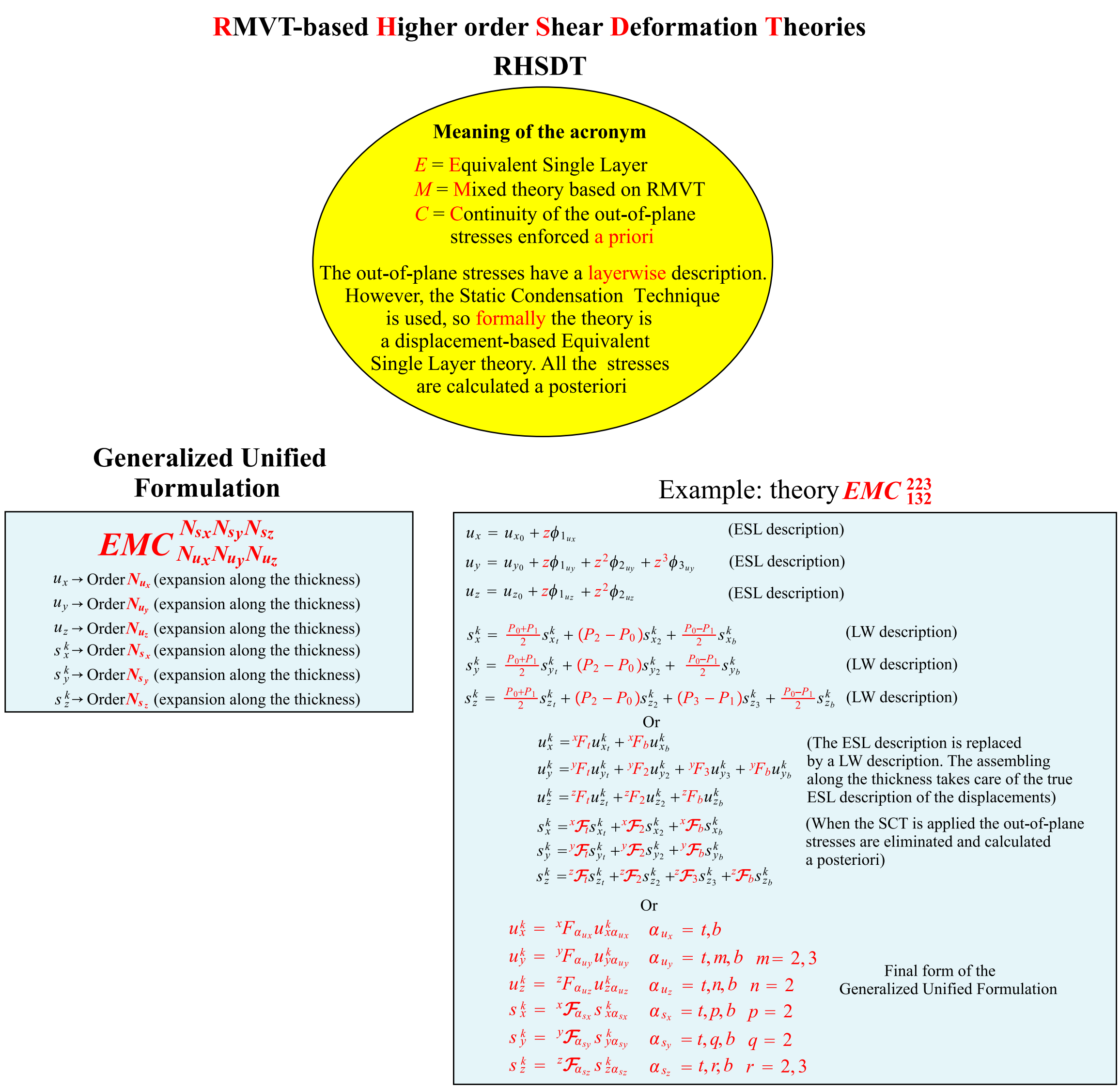

Fig. 2. Acronyms used in the case of Reissner's Mixed Variational Theorem-based higher order shear deformation theories (RHSDT).

was then formulated for the case of Reissener's Mixed Variational Theorem (see references [17,18]) in *Part I* (reference [19]). GUF was generalized to the case of layerwise theories (see reference [20]). This work extends the Generalized Unified Formulation to the case of RMVT-based higher order shear deformation theories. *Any* combination of orders used for the expansion of the displacements and out-of-plane stresses in the thickness direction is possible. Therefore, with this formalism ∞^6 *new* mixed higher order shear deformation theories are introduced for the first time in the literature. *All the theories* are generated by expanding thirteen fundamental invariant kernels which are the same as the ones introduced in *Part I* and used in *Part II* in the case of layerwise theories.

2. Theoretical derivation of ∞^6 mixed equivalent single layer theories

The derivation is started by considering a particular theory in which the in-plane displacements are expanded along the thickness by using a cubic polynomial and the out-of-plane displacement u_z is parabolic. In this case it is possible to write the displacements as follows:

$$\text{Theory I}: \begin{cases} u_x = u_{x_0} + z\phi_{1_{u_x}} + z^2\phi_{2_{u_x}} + z^3\phi_{3_{u_x}} \\ u_y = u_{y_0} + z\phi_{1_{u_y}} + z^2\phi_{2_{u_y}} + z^3\phi_{3_{u_y}} \\ u_z = u_{z_0} + z\phi_{1_{u_z}} + z^2\phi_{2_{u_z}} \end{cases} \tag{4}$$

For *each displacement component* the concepts of the Generalized Unified Formulation (see *Part I* [19]) can be applied. For example, the displacement u_x is written as

$$\begin{aligned} u_x &= u_{x_0} + z\phi_{1_{u_x}} + z^2\phi_{2_{u_x}} + z^3\phi_{3_{u_x}} \\ &= {}^xF_t u_{x_t} + {}^xF_2 u_{x_2} + {}^xF_3 u_{x_3} + {}^xF_b u_{x_b} \end{aligned} \tag{5}$$

where

$$\begin{aligned} &{}^xF_t = 1; \quad {}^xF_2 = z; \quad {}^xF_3 = z^2; \quad {}^xF_b = z^3 \quad u_{x_t} = u_{x_0}; \\ &u_{x_2} = \phi_{1_{u_x}}; \quad u_{x_3} = \phi_{2_{u_x}}; \quad u_{x_b} = \phi_{3_{u_x}} \end{aligned} \tag{6}$$

The GUF for the displacement u_x is

$$u_x = {}^xF_{\alpha_{u_x}} u_{x\alpha_{u_x}} \quad \alpha_{u_x} = t, l, b; \quad l = 2, \dots, N_{u_x} \tag{7}$$

where, in the example, $N_{u_x} = 3$. Notice that the superscript "x" in ${}^xF_{\alpha_{u_x}}$ is used to clearly enhance that the displacement u_x (displacement in the x direction) is being considered. For the displacement u_y

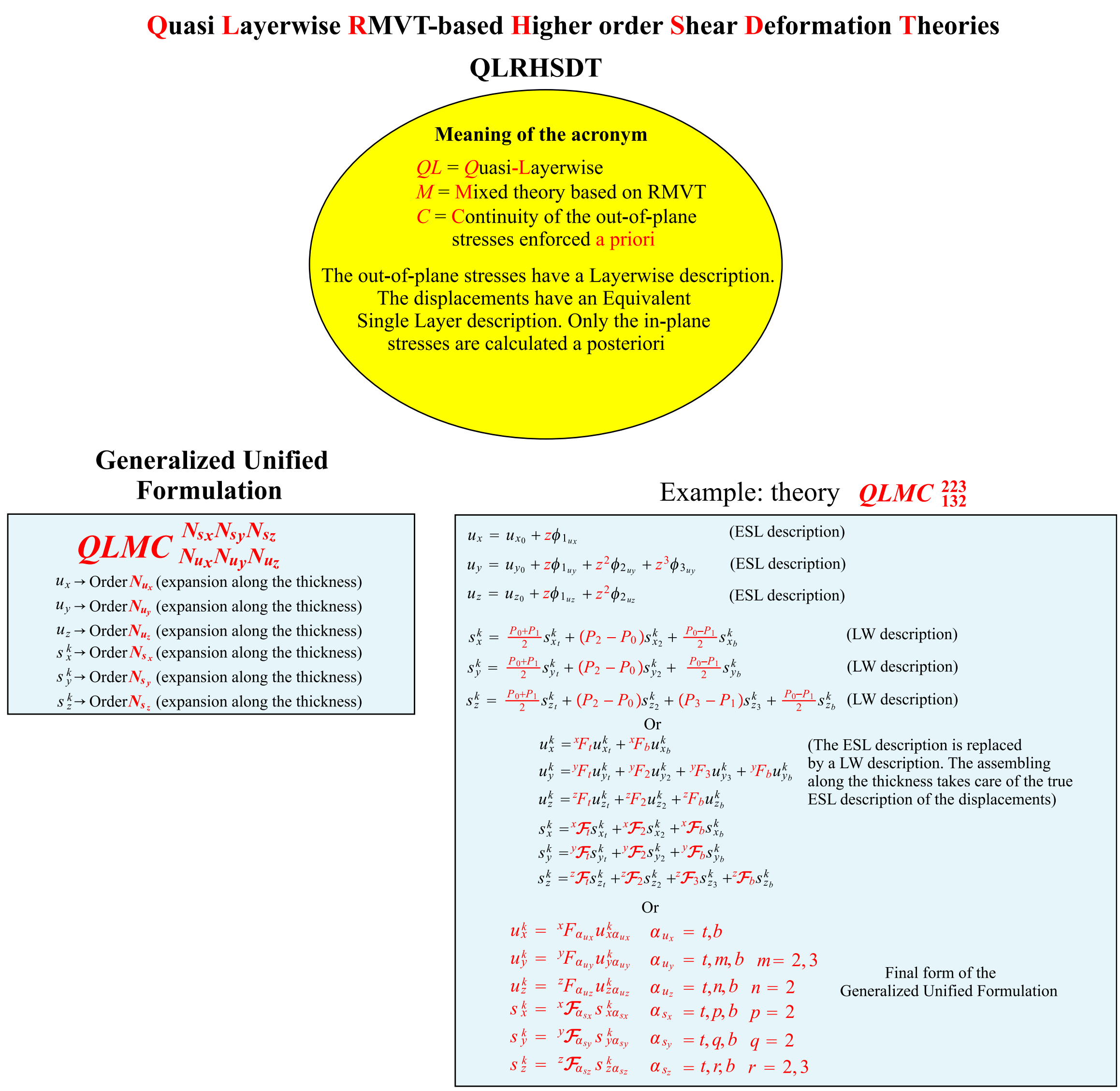

Fig. 3. Acronyms used in the case of Quasi-Layerwise Reissner's Mixed Variational Teorem-based higher order shear deformation theories (QLRHSDT).

Please cite this article in press as: Demasi L, ∞^6 Mixed plate theories based on the Generalized Unified Formulation. ..., Compos Struct (2008), doi:10.1016/j.compstruct.2008.07.011

the formal procedure produces similar results (notice that now the slave index is u_y):

$$u_y = {}^yF_{\alpha_{u_y}} u_{y\alpha_{u_y}} \quad \alpha_{u_y} = t, m, b; \quad m = 2, \ldots, N_{u_y} \tag{8}$$

The displacement u_z is only parabolic (three terms are used in the expansion), but the representation is formally the same:

$$u_z = {}^zF_{\alpha_{u_z}} u_{z\alpha_{u_z}} \quad \alpha_{u_z} = t, n, b; \quad n = 2, \ldots, N_{u_z} \tag{9}$$

The following are observed:

- The superscript k is not present. In fact, in equivalent single layer models the displacement fields have a description at plate level and not at layerwise level.
- The Generalized Unified Formulation (Eqs. (7)–(9)) is formally equivalent to the layerwise case (see *Part I* and *Part II* in references [19,20]). The layerwise GUF is

$$\begin{aligned} u_x^k &= {}^xF_{\alpha_{u_x}} u^k_{x\alpha_{u_x}} \quad \alpha_{u_x} = t, l, b; \quad l = 2, \ldots, N_{u_x} \\ u_y^k &= {}^yF_{\alpha_{u_y}} u^k_{y\alpha_{u_y}} \quad \alpha_{u_y} = t, m, b; \quad m = 2, \ldots, N_{u_y} \\ u_z^k &= {}^zF_{\alpha_{u_z}} u^k_{z\alpha_{u_z}} \quad \alpha_{u_z} = t, n, b; \quad n = 2, \ldots, N_{u_z} \end{aligned} \tag{10}$$

The similarity between the equivalent single layer (ESL) and the layerwise cases suggests that it is possible to use the layerwise Generalized Unified Formulation for the ESL case too. That is, Eq. (10) *can be used for the equivalent single layer case*. The fact that the displacement field *does not* have a layerwise description (see, for example, Theory I explicitly written in Eq. (4)) is taken into account when the *assembling in the thickness direction* of the layer matrices is considered.

A *mixed* formulation is going to be used. That is, the modeled unknowns are *not* just the displacements. Some of the stresses are modeled as well. The continuity of the out-of-plane stresses is enforced by using a *layerwise* description. This is not in contradiction with the fact that an equivalent single layer category of theories is going to be developed. In fact, the stresses are *eliminated* by using the Static Condensation Technique as explained in *Part I* (see [19]). This operation is important when a FEM approach is considered. The static condensation at element level can reduce the CPU time significantly. The following terminology is introduced here:

- *Reissner's Mixed Variational Theorem-based higher order shear deformation theories (RHSDT)*
  These theories have an equivalent single layer description of the displacement fields and a layerwise description of the out-of-plane stresses. However, the stresses are eliminated using the Static Condensation Technique (see *Part I*). Therefore, formally, the theories have a final equivalent single layer description as the "classical" displacement-based theories. This approach gives, for example, a theory similar to Theory I in which the stresses are *formally* not assumed (displacement-based theory).
- *Quasi-Layerwise Reissner's Mixed Variational Theorem-based higher order shear deformation theories (QLRHSDT)*
  These theories have an equivalent single layer description of the displacement fields and a layerwise description of the out-of-plane stresses. The static condensation *is not* used. Thus, this type of theories describes the displacements as in the ESL approach and the stresses σ_{xz}, σ_{yz}, σ_{zz} as layerwise quantities.

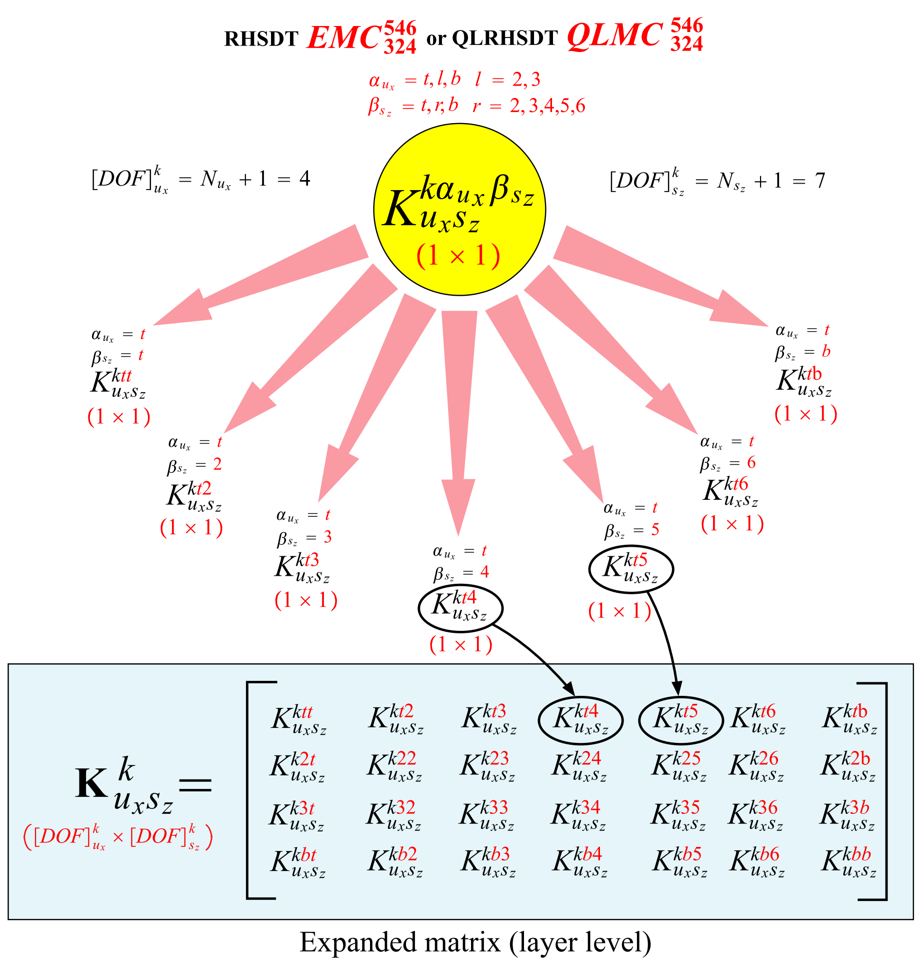

Fig. 4. Generalized Unified Formulation: example of expansion from a kernel to a layer matrix. The case of theories EMC^{546}_{324} and $QLMC^{546}_{324}$. From $K^{k\alpha_{u_x}\beta_{s_z}}_{u_x s_z}$ to $\mathbf{K}^k_{u_x s_z}$.

A Navier-type solution will be considered in this paper. Therefore, considering that the static condensation technique (SCT) is applied at plate level, the difference between RHSDT and QLRHSDT is practically only formal. The results will be coincident. In the FEM codes this will not be true since the SCT will, conveniently, be applied at element level. The Generalized Unified Formulation valid for both RHSDT and QLRHSDT is then the following (the simplified notation for the stresses used is $\sigma_{xz}^k = s_x^k$, $\sigma_{yz}^k = s_y^k$, $\sigma_{zz}^k = s_z^k$):

$$
\begin{aligned}
&u_x^k = {}^xF_t u_{x_t}^k + {}^xF_l u_{x_l}^k + {}^xF_b u_{x_b}^k = {}^xF_{\alpha_{u_x}} u_{x\alpha_{u_x}}^k \\
&\alpha_{u_x} = t, l, b; \quad l = 2, \ldots, N_{u_x} \\
&u_y^k = {}^yF_t u_{y_t}^k + {}^yF_m u_{y_m}^k + {}^yF_b u_{y_b}^k = {}^yF_{\alpha_{u_y}} u_{y\alpha_{u_y}}^k \\
&\alpha_{u_y} = t, m, b; \quad m = 2, \ldots, N_{u_y} \\
&u_z^k = {}^zF_t u_{z_t}^k + {}^zF_n u_{z_n}^k + {}^zF_b u_{z_b}^k = {}^zF_{\alpha_{u_z}} u_{z\alpha_{u_z}}^k \\
&\alpha_{u_z} = t, n, b; \quad n = 2, \ldots, N_{u_z} \\
&s_x^k = {}^x\mathcal{F}_t s_{x_t}^k + {}^x\mathcal{F}_p s_{x_p}^k + {}^x\mathcal{F}_b s_{x_b}^k = {}^x\mathcal{F}_{\alpha_{s_x}} s_{x\alpha_{s_x}}^k \\
&\alpha_{s_x} = t, p, b; \quad p = 2, \ldots, N_{s_x} \\
&s_y^k = {}^y\mathcal{F}_t s_{y_t}^k + {}^y\mathcal{F}_q s_{y_q}^k + {}^y\mathcal{F}_b s_{y_b}^k = {}^y\mathcal{F}_{\alpha_{s_y}} s_{y\alpha_{s_y}}^k \\
&\alpha_{s_y} = t, q, b; \quad q = 2, \ldots, N_{s_y} \\
&s_z^k = {}^z\mathcal{F}_t s_{z_t}^k + {}^z\mathcal{F}_r s_{z_r}^k + {}^z\mathcal{F}_b s_{z_b}^k = {}^z\mathcal{F}_{\alpha_{s_z}} s_{z\alpha_{s_z}}^k \\
&\alpha_{s_z} = t, r, b; \quad r = 2, \ldots, N_{s_z}
\end{aligned} \tag{11}
$$

The functions of the thickness coordinate are assumed to be of the type $1\ z\ z^2\ z^3 \ldots$. This choice is made for "consistency" with the usual approach used in the literature for equivalent single layer models. However, the out-of-plane stresses are modeled as layerwise quantities. A combination of Legendre polynomials as explained in the layerwise models (see *Part II*, reference [20]) is then used. In detail, the functions used in the expansions of the displacements are:

$$
\begin{aligned}
&{}^xF_t = 1 \quad {}^yF_t = 1 \quad {}^zF_t = 1 \\
&{}^xF_2 = z \quad {}^yF_2 = z \quad {}^zF_2 = z \\
&{}^xF_3 = z^2 \quad {}^yF_3 = z^2 \quad {}^zF_3 = z^2 \\
&\ldots \\
&{}^xF_l = z^{l-1} \quad {}^yF_m = z^{m-1} \quad {}^zF_n = z^{n-1} \\
&\ldots \\
&{}^xF_b = z^{N_{u_x}} \quad {}^yF_b = z^{N_{u_y}} \quad {}^zF_b = z^{N_{u_z}}
\end{aligned} \tag{12}
$$

When a particular layer k is considered the z is contained in the interval $[z_{\mathrm{bot}_k}, z_{\mathrm{top}_k}]$. That is,

$$
z_{\mathrm{bot}_k} \leqslant z \leqslant z_{\mathrm{top}_k} \tag{13}
$$

where z_{bot_k} is the thickness coordinate of the *bottom surface of layer k* and z_{top_k} is the thickness coordinate of the *top surface of layer k*. The origin of the coordinate system is in the middle plane of the plate (see Fig. 1 of reference [19]). For the out-of-plane stresses a

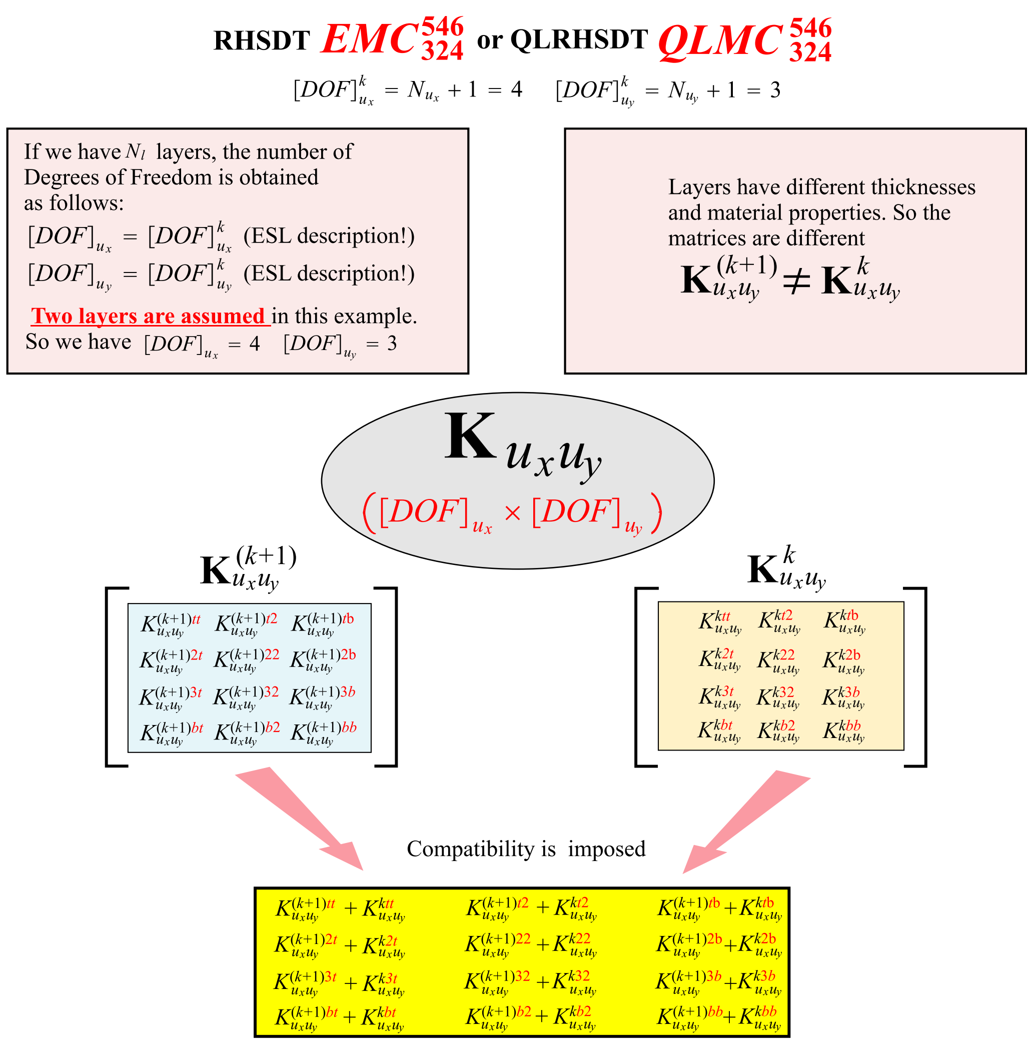

Fig. 5. Generalized Unified Formulation: example of assembling from layer matrices to multilayer matrix. Case of theory EMC_{324}^{546} or $QLMC_{324}^{546}$. From $\mathbf{K}_{u_x u_y}^k$ and $\mathbf{K}_{u_x u_y}^{(k+1)}$ to $\mathbf{K}_{u_x u_y}$.

combination of Legendre polynomials is used. The polynomials are defined in the interval $[-1,+1]$. Therefore, a transformation of coordinate is required:

$$\zeta_k = \frac{2}{z_{top_k} - z_{bot_k}} z - \frac{z_{top_k} + z_{bot_k}}{z_{top_k} - z_{bot_k}} \tag{14}$$

z is measured from the middle plane of the plate. For example, consider a plate with thickness h and two layers of equal thickness. In this case when the top layer is considered ($k = 2$) it is possible to write:

$$z_{top_{k=2}} = \frac{h}{2} \quad z_{bot_{k=2}} = 0 \tag{15}$$

Similarly, for the bottom layer ($k = 1$):

$$z_{top_{k=1}} = 0 \quad z_{bot_{k=1}} = -\frac{h}{2} \tag{16}$$

In the example of a plate with two layers of equal thickness the transformation of coordinate (Eq. 14) is

$$\begin{aligned} &\text{layer } 1 \Rightarrow \zeta_{k=1} = \frac{4}{h} z + 1 \\ &\text{layer } 2 \Rightarrow \zeta_{k=2} = \frac{4}{h} z - 1 \end{aligned} \tag{17}$$

For the expansion of the stresses $\sigma_{xz} = s_x^k$, $\sigma_{yz} = s_y^k$ and $\sigma_{zz} = s_z^k$ the functions used for the expansions along the thickness are the following:

$$\begin{aligned} &{}^x\mathcal{F}_t = {}^y\mathcal{F}_t = {}^z\mathcal{F}_t = \frac{P_0 + P_1}{2}, \quad {}^x\mathcal{F}_b = {}^y\mathcal{F}_b = {}^z\mathcal{F}_b = \frac{P_0 - P_1}{2} \\ &{}^x\mathcal{F}_p = P_p - P_{p-2}, \quad p = 2, 3, \ldots, N_{s_x} \\ &{}^y\mathcal{F}_q = P_q - P_{q-2}, \quad q = 2, 3, \ldots, N_{s_y} \\ &{}^z\mathcal{F}_r = P_r - P_{r-2}, \quad r = 2, 3, \ldots, N_{s_z} \end{aligned} \tag{18}$$

where P_0, P_1, P_p, P_q, P_r are the Legendre polynomials of order 0, 1, p, q and r respectively. More details on these polynomials can be found in *Part II* (see reference [20]).

With combination of Legendre polynomials, the functions used for the expansions of the stress variables have useful (for the assembling in the thickness direction, see *Part II*) properties:

$$\zeta_k = \begin{cases} +1, & {}^x\mathcal{F}_t, {}^y\mathcal{F}_t, {}^z\mathcal{F}_t = 1, \\ {}^x\mathcal{F}_b, {}^y\mathcal{F}_b, {}^z\mathcal{F}_b = 0, & {}^x\mathcal{F}_l, {}^y\mathcal{F}_m, {}^z\mathcal{F}_n = 0, \\ -1, & {}^x\mathcal{F}_t, {}^y\mathcal{F}_t, {}^z\mathcal{F}_t = 0, \\ {}^x\mathcal{F}_b, {}^y\mathcal{F}_b, {}^z\mathcal{F}_b = 1, & {}^x\mathcal{F}_l, {}^y\mathcal{F}_m, {}^z\mathcal{F}_n = 0. \end{cases} \tag{19}$$

Reissner's Mixed Variational Theorem-based higher order shear deformation theories (RHSDT) will be indicated with acronyms shown in Fig. 2.

The Quasi-Layerwise Reissner's Mixed Variational Theorem-based higher order shear deformation theories (QLRHSDT) will be indicated with acronyms shown in Fig. 3.

In both RHSDT and QLRHSDT cases it is then possible to create a class of theories by changing the order used for the displacements and stresses. Suppose, for example, that a theory has the following data: $N_{u_x} = 3$, $N_{u_y} = 2$, $N_{u_z} = 4$, $N_{s_x} = 5$, $N_{s_y} = 4$, $N_{s_z} = 6$. The corresponding RHSDT theory is indicated as EMC_{324}^{546}. The first letter "*E*" means "equivalent single layer" theory, the second letter "*M*" means that a mixed variational statement is used (Reissner's Variational Theorem) and the third letter "*C*" means that the continuity of the out-of-plane stresses is enforced a priori. The subscripts are the orders of the polynomials used for the displacements. The superscripts are the orders of the Legendre polynomials used for the out-of-plane stresses. In general, the acronym is then built as follows: $EMC_{N_{u_x}N_{u_y}N_{u_z}}^{N_{s_x}N_{s_y}N_{s_z}}$.Similarly, with the same orders for displacements and stresses it is possible to build QLRHSDT theories. These theories are indicated with the acronyms $QLMC_{N_{u_x}N_{u_y}N_{u_z}}^{N_{s_x}N_{s_y}N_{s_z}}$. "*Q*" stands for "Quasi" and "*L*" stands for "Layerwise". In this paper there is no numerical difference between a generic theory $EMC_{N_{u_x}N_{u_y}N_{u_z}}^{N_{s_x}N_{s_y}N_{s_z}}$ and the

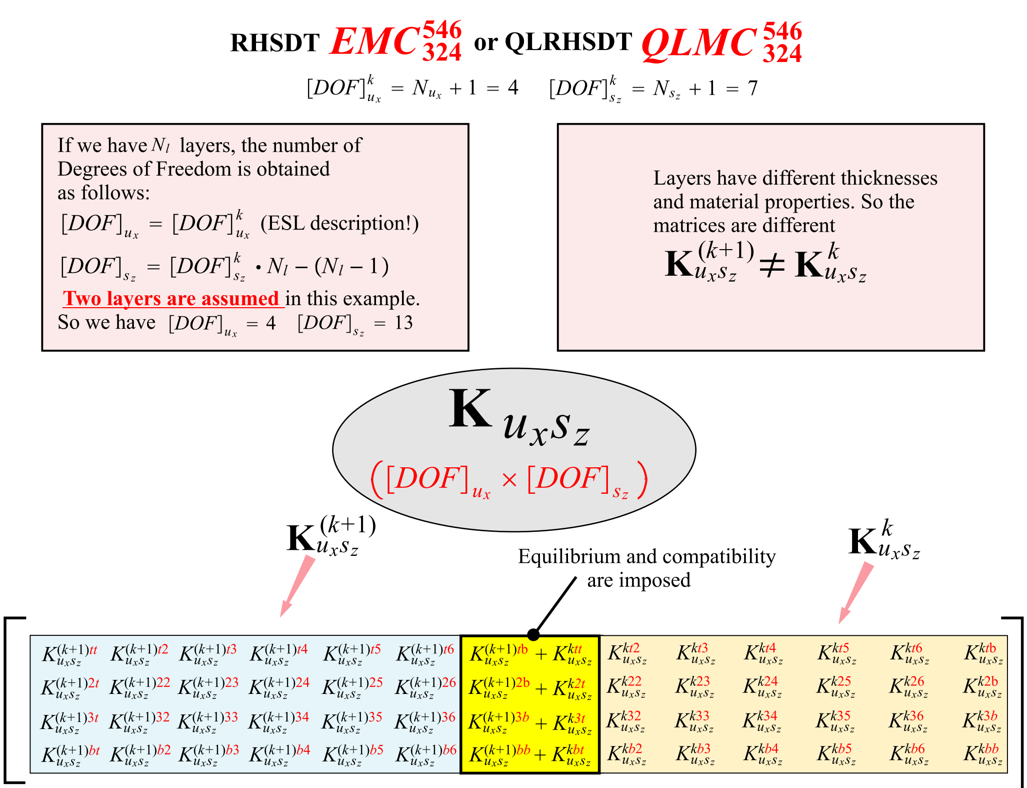

Fig. 6. Generalized Unified Formulation: example of assembling from layer matrices to multilayer matrix. Case of theory EMC_{324}^{546} or $QLMC_{324}^{546}$. From $\mathbf{K}_{u_x s_z}^k$ and $\mathbf{K}_{u_x s_z}^{(k+1)}$ to $\mathbf{K}_{u_x s_z}$.

Please cite this article in press as: Demasi L, ∞^6 Mixed plate theories based on the Generalized Unified Formulation. ..., Compos Struct (2008), doi:10.1016/j.compstruct.2008.07.011

corresponding $QLMC^{N_{s_x}N_{s_y}N_{s_z}}_{N_{u_x}N_{u_y}N_{u_z}}$ theory. The reason, as stated before, is that the Static Condensation Technique is performed at plate level. But when FEM applications are considered this is no longer true, since it is convenient to perform the static condensation at element level and so the continuity in the plane x–y of the out-of-plane stresses is not enforced a priori.

2.1. Expansion of the matrices

This is the most important part of the generation of one of the ∞^6 theories. This operation is done *at layer level*. To explain how this operation is performed, consider theory EMC^{546}_{324} (theory $QLMC^{546}_{324}$ is equivalent at this point). When (see *Part II*, reference [20]) layerwise theories were discussed the case of theory LM^{546}_{324} was analyzed. This is the reason why a case with the same orders of expansion is considered. In the case of theory EMC^{546}_{324} the number of degrees of freedom (at *layer level*) is the following:

$$[DOF]^k_{u_x} = N_{u_x} + 1 = 3 + 1 = 4 \quad [DOF]^k_{u_y} = N_{u_y} + 1 = 2 + 1 = 3$$
$$[DOF]^k_{u_z} = N_{u_z} + 1 = 4 + 1 = 5 \quad [DOF]^k_{s_x} = N_{s_x} + 1 = 5 + 1 = 6$$
$$[DOF]^k_{s_y} = N_{s_y} + 1 = 4 + 1 = 5 \quad [DOF]^k_{s_z} = N_{s_z} + 1 = 6 + 1 = 7 \tag{20}$$

From the number of degrees of freedom the size of the layer matrices can be calculated. For example, when the matrix $K^{k\alpha_{u_x}\beta_{s_z}}_{u_x s_z}$ is expanded then the final size at layer level will be $[DOF]^k_{u_x} \times [DOF]^k_{s_z}$. In the example relative to theory EMC^{546}_{324} the matrix $K^{k\alpha_{u_x}\beta_{s_z}}_{u_x s_z}$ at layer level (indicated as $\mathbf{K}^k_{u_x s_z}$) is obtained as explained in Fig. 4. It should be noticed that at this point there is no difference between the case of RHSDT and QLRHSDT: Fig. 4 is valid for both cases.

2.2. Assembling in the thickness direction

In addition to the compatibility of the displacements, the equilibrium between two adjacent layers implies that $s^k_{x_t} = s^{(k+1)}_{x_b}$, $s^k_{y_t} = s^{(k+1)}_{y_b}$ and $s^k_{z_t} = s^{(k+1)}_{z_b}$ (see Fig. 4 in *Part II*). Therefore, the assembling must consider this fact. Also, the displacement fields are treated as equivalent single layers quantities, and this makes the assembling procedure different with respect to the layerwise case seen in *Part II*.

Regarding thickness assembling, there can be different cases:

- *Case 1*
  It involves only the displacement degrees of freedom. This is, for example, the case when the multilayer matrix $\mathbf{K}_{u_x u_y}$ is generated. The assembling must take into account the continuity of the displacements and their ESL description (see Fig. 5).

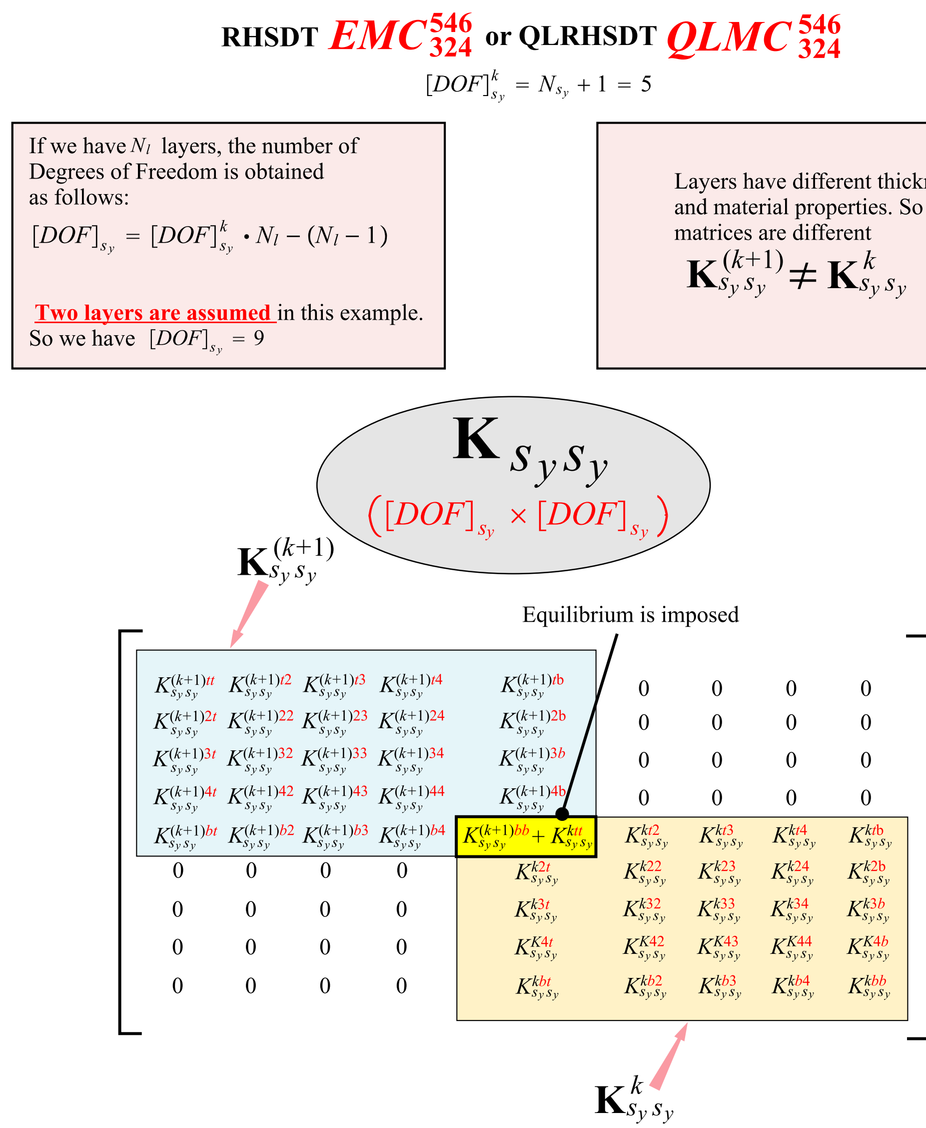

Fig. 7. Generalized Unified Formulation: example of assembling from layer matrices to multilayer matrix. Case of theory EMC^{546}_{324} or $QLMC^{546}_{324}$. From $\mathbf{K}^k_{s_y s_y}$ and $\mathbf{K}^{(k+1)}_{s_y s_y}$ to $\mathbf{K}_{s_y s_y}$.

Please cite this article in press as: Demasi L, ∞^6 Mixed plate theories based on the Generalized Unified Formulation. ..., Compos Struct (2008), doi:10.1016/j.compstruct.2008.07.011

- *Case 2*
  It involves displacement and out-of-plane stress degrees of freedom. This is, for example, the case when the multilayer matrix $\mathbf{K}_{u_x s_z}$ is generated. The assembling must take into account the continuity of the displacements and their ESL description. In addition, the equilibrium of the transverse stresses must be enforced (see Fig. 6).
- *Case 3*
  It involves only the out-of-plane stress degrees of freedom. This is, for example, the case when the multilayer matrix $\mathbf{K}_{s_y s_y}$ is generated. The assembling must take into account the equilibrium of the transverse stresses (see Fig. 7). When only the out-of-plane stress degrees of freedom are involved a pure layerwise description is used. In fact, Fig. 7 is formally similar to Figure 5 of Part II (reference [20]) which is used in the layerwise case.

The pressure matrices are obtained from the pressure kernels using the same formal method shown in Fig. 4. The sizes of the layer matrices ${}^t\mathbf{D}^k_{u_x u_x}$ and ${}^b\mathbf{D}^k_{u_x u_x}$ are the same as the size of matrix $\mathbf{K}^k_{u_x u_x}$. Similarly, the sizes of the matrices ${}^t\mathbf{D}^k_{u_y u_y}$ and ${}^b\mathbf{D}^k_{u_y u_y}$ are the same as the size of matrix $\mathbf{K}^k_{u_y u_y}$. Finally, the sizes of matrices ${}^t\mathbf{D}^k_{u_z u_z}$ and ${}^b\mathbf{D}^k_{u_z u_z}$ are the same as the size of matrix $\mathbf{K}^k_{u_z u_z}$. The pressures can be applied only at the top or bottom surfaces of the plate. This means that the pressure matrices at layer level are calculated only for $k = N_l$ and $k = 1$, the top and bottom layers respectively. In particular, ${}^t\mathbf{D}^k_{u_x u_x}$, ${}^t\mathbf{D}^k_{u_y u_y}$ and ${}^t\mathbf{D}^k_{u_z u_z}$ are calculated only for $k = N_l$ (for the other layers these matrices are set to be with only zeros). Similarly, ${}^b\mathbf{D}^k_{u_x u_x}$, ${}^b\mathbf{D}^k_{u_y u_y}$ and ${}^b\mathbf{D}^k_{u_z u_z}$ are calculated only for $k = 1$ (for the other layers these matrices are set to be with only zeros). The assembling to multilayer level is then done as for the corresponding matrices. For example, matrix ${}^t\mathbf{D}_{u_x u_x}$ is built using the same procedure used for matrix $\mathbf{K}_{u_x u_x}$. About the pressure amplitudes, inputs of the problem (see Fig. 8), the difference between this ESL case and the corresponding layerwise case seen in *Part II* (reference [20]) should be noted.

Fig. 9 shows the amplitude vectors for a particular case.

The pressure amplitudes at multilayer level are *inputs* of the problem. Some input examples are shown in Fig. 8. Once the matrices are all assembled, the system of equations becomes (see *Part I* for details on the derivation):

$$\begin{bmatrix} \mathbf{K}_{u_x u_x} & \mathbf{K}_{u_x u_y} & \mathbf{0}_{u_x u_z} & \mathbf{K}_{u_x s_x} & \mathbf{0}_{u_x s_y} & \mathbf{K}_{u_x s_z} \\ & \mathbf{K}_{u_y u_y} & \mathbf{0}_{u_y u_z} & \mathbf{0}_{u_y s_x} & \mathbf{K}_{u_y s_y} & \mathbf{K}_{u_y s_z} \\ & & \mathbf{0}_{u_z u_z} & \mathbf{K}_{u_z s_x} & \mathbf{K}_{u_z s_y} & \mathbf{K}_{u_z s_z} \\ & & & \mathbf{K}_{s_x s_x} & \mathbf{0}_{s_x s_y} & \mathbf{0}_{s_x s_z} \\ \text{Symm} & & & & \mathbf{K}_{s_y s_y} & \mathbf{0}_{s_y s_z} \\ & & & & & \mathbf{K}_{s_z s_z} \end{bmatrix} \cdot \begin{bmatrix} {}^x\mathbf{U} \\ {}^y\mathbf{U} \\ {}^z\mathbf{U} \\ {}^x\mathbf{S} \\ {}^y\mathbf{S} \\ {}^z\mathbf{S} \end{bmatrix} = \begin{bmatrix} {}^x\mathbf{R} \\ {}^y\mathbf{R} \\ {}^z\mathbf{R} \\ {}^x\mathbf{0} \\ {}^y\mathbf{0} \\ {}^z\mathbf{0} \end{bmatrix} \tag{21}$$

where

$$\begin{aligned} {}^x\mathbf{R} &= {}^t\mathbf{D}_{u_x u_x} \cdot {}^x\mathbf{P}^t + {}^b\mathbf{D}_{u_x u_x} \cdot {}^x\mathbf{P}^b \\ {}^y\mathbf{R} &= {}^t\mathbf{D}_{u_y u_y} \cdot {}^y\mathbf{P}^t + {}^b\mathbf{D}_{u_y u_y} \cdot {}^y\mathbf{P}^b \\ {}^z\mathbf{R} &= {}^t\mathbf{D}_{u_z u_z} \cdot {}^z\mathbf{P}^t + {}^b\mathbf{D}_{u_z u_z} \cdot {}^z\mathbf{P}^b \end{aligned} \tag{22}$$

These expressions are formally the same as the ones encountered in the layerwise case. This equivalence is another advantage of the Generalized Unified Formulation.

3. Example: a multilayered plate

To show how the theories are created, consider a rectangular multilayered plate with two layers. Let the thickness of the plate be h. The bottom layer has thickness $h_{k=1} = \frac{4}{7}h$. The top layer has thickness $h_{k=2} = \frac{3}{7}h$. How the matrices are generated from the kernels of the Generalized Unified Formulation is shown. Consider the

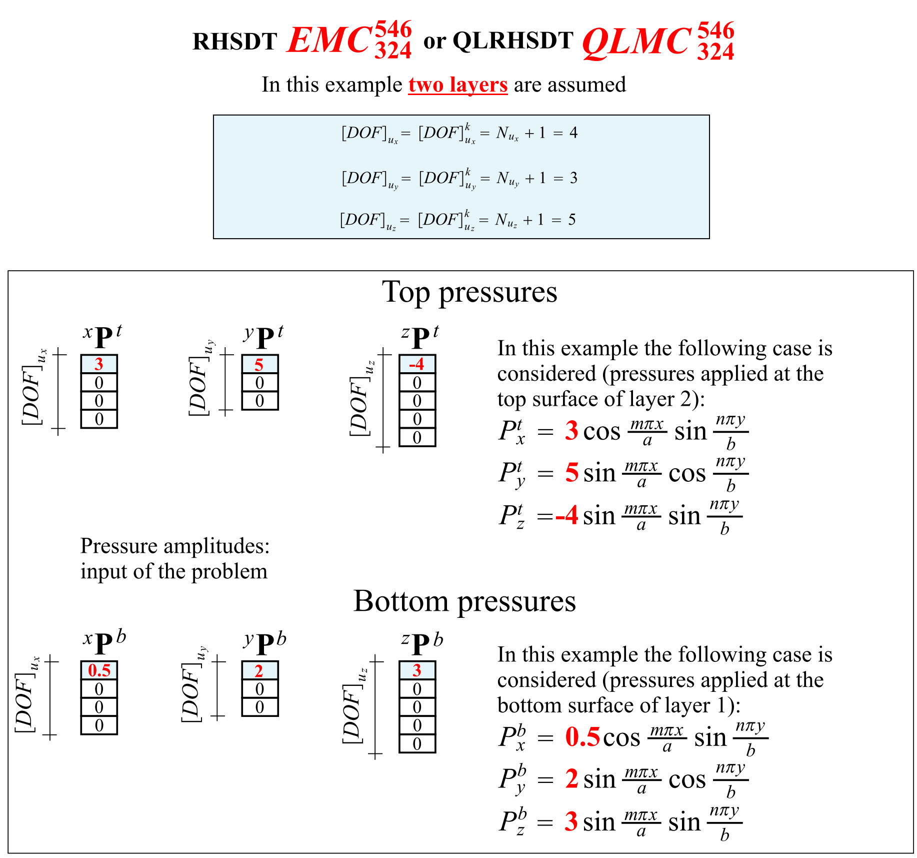

Fig. 8. Case of theory EMC^{546}_{324} or $QLMC^{546}_{324}$. Example of pressure amplitudes and inputs at multilayer level for the case in two-layered case.

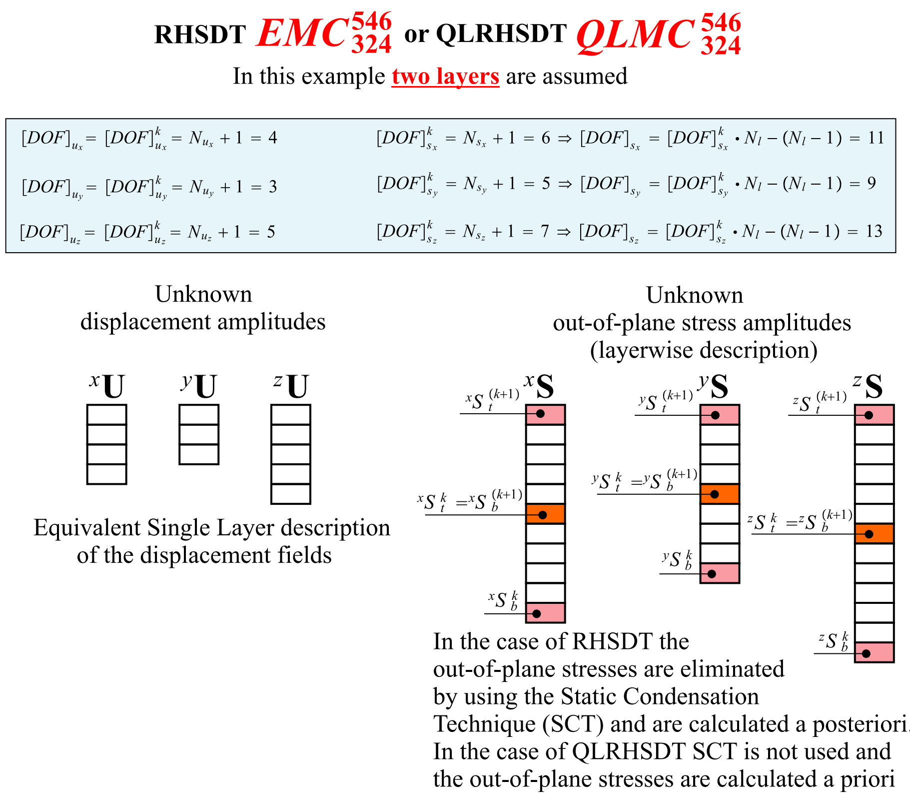

Fig. 9. Case of theories EMC^{546}_{324} and $QLMC^{546}_{324}$. Multilayer unknown displacements and out-of-plane vectors in two-layered case.

top layer matrix $\mathbf{K}^{k=2}_{u_x s_z}$. The *kernel* associated with this matrix at layer level is the following (see *Part I*):

$$K^{k=2\,\alpha_{u_x}\,\beta_{s_z}}_{u_x s_z} = -\frac{m\pi}{a} Z^{k=2\,\alpha_{u_x}\,\beta_{s_z}}_{13 u_x s_z} = -\frac{m\pi}{a} C^{k=2}_{13} \int_{z_{bot_{k=2}}}^{z_{top_{k=2}}} {}^xF_{\alpha_{u_x}}(z)\,{}^z\mathcal{F}_{\alpha_{s_z}}(z)\,dz \tag{23}$$

where $z_{bot_{k=2}}$ is the z coordinate of the bottom surface of layer $k = 2$ (top layer); $z_{top_{k=2}}$ is the z coordinate of the top surface of layer $k = 2$. The reference plane is the middle plane of the whole plate. Thus, for the top layer:

$$z_{bot_{k=2}} = \frac{h}{14} \quad z_{top_{k=2}} = \frac{h}{2} \tag{24}$$

The expression of the kernel is then

$$K^{k=2\,\alpha_{u_x}\,\beta_{s_z}}_{u_x s_z} = -\frac{m\pi}{a} C^{k=2}_{13} \int_{\frac{h}{14}}^{\frac{h}{2}} {}^xF_{\alpha_{u_x}}(z)\,{}^z\mathcal{F}_{\alpha_{s_z}}(z)\,dz \quad \frac{h}{14} \leqslant z \leqslant \frac{h}{2} \tag{25}$$

Consider again theory EMC^{546}_{324} and, in particular, the term in which $\alpha_{u_x} = 3$ and $\beta_{s_z} = 2$. In this case Eq. (25) becomes:

$$K^{k=2\,32}_{u_x s_z} = -\frac{m\pi}{a} C^{k=2}_{13} \int_{\frac{h}{14}}^{\frac{h}{2}} {}^xF_3(z)\,{}^z\mathcal{F}_2(z)\,dz \tag{26}$$

In practice it is more convenient to transform the variables (see Eq. (14)) and numerically integrate using Gauss's quadrature formula (see reference [21]) in the interval $[-1,+1]$. However, since the goal is to show the procedure, for simplicity the physical coordinate z is used. It is the possible to write:

$$ {}^xF_3(z) = z^2 \quad {}^z\mathcal{F}_2(z) = P_2(z) - P_0(z) = \frac{3(\zeta_{k=2})^2 - 1}{2} - 1 = \frac{3\left(\frac{2}{z_{top_{k=2}} - z_{bot_{k=2}}} z - \frac{z_{top_{k=2}} + z_{bot_{k=2}}}{z_{top_{k=2}} - z_{bot_{k=2}}}\right)^2 - 1}{2} - 1 \tag{27}$$

Using Eq. (24)

$${}^xF_3(z) = z^2 \quad {}^z\mathcal{F}_2(z) = \frac{7(h - 14z)(h - 2z)}{6h^2} \tag{28}$$

Substituting into Eq. (26):

$$K^{k=2\,32}_{u_x s_z} = -\frac{m\pi}{a} C^{k=2}_{13} \int_{\frac{h}{14}}^{\frac{h}{2}} z^2 \cdot \frac{7(h - 14z)(h - 2z)}{6h^2}\,dz = \frac{h^3 m\pi}{a} C^{k=2}_{13} \frac{267}{6860} \tag{29}$$

Using the same procedure, it is possible to demonstrate that the top layer (with the above mentioned data) has matrices reported in Appendix A.

The pressure matrices are obtained by using the definitions reported in *Part I*. For example, for the top layer ($k = 2$):

$${}^tD^{k=2\,\alpha_{u_x}\,\beta_{u_x}}_{u_x u_x} = {}^xF^t_{\alpha_{u_x}}\,{}^xF^t_{\beta_{u_x}} = {}^xF_{\alpha_{u_x}}\left(z = +\frac{h}{2}\right)\,{}^xF_{\beta_{u_x}}\left(z = +\frac{h}{2}\right) \tag{30}$$

Therefore, the pressure matrices are the following:

$${}^t\mathbf{D}^{k=2}_{u_x u_x} = \begin{bmatrix} 1 & \frac{h}{2} & \frac{h^2}{4} & \frac{h^3}{8} \\ \frac{h}{2} & \frac{h^2}{4} & \frac{h^3}{8} & \frac{h^4}{16} \\ \frac{h^2}{4} & \frac{h^3}{8} & \frac{h^4}{16} & \frac{h^5}{32} \\ \frac{h^3}{8} & \frac{h^4}{16} & \frac{h^5}{32} & \frac{h^6}{64} \end{bmatrix} \tag{31}$$

$${}^t\mathbf{D}^{k=2}_{u_y u_y} = \begin{bmatrix} 1 & \frac{h}{2} & \frac{h^2}{4} \\ \frac{h}{2} & \frac{h^2}{4} & \frac{h^3}{8} \\ \frac{h^2}{4} & \frac{h^3}{8} & \frac{h^4}{16} \end{bmatrix} \tag{32}$$

$${}^t\mathbf{D}^{k=2}_{u_z u_z} = \begin{bmatrix} 1 & \frac{h}{2} & \frac{h^2}{4} & \frac{h^3}{8} & \frac{h^4}{16} \\ \frac{h}{2} & \frac{h^2}{4} & \frac{h^3}{8} & \frac{h^4}{16} & \frac{h^5}{32} \\ \frac{h^2}{4} & \frac{h^3}{8} & \frac{h^4}{16} & \frac{h^5}{32} & \frac{h^6}{64} \\ \frac{h^3}{8} & \frac{h^4}{16} & \frac{h^5}{32} & \frac{h^6}{64} & \frac{h^7}{128} \\ \frac{h^4}{16} & \frac{h^5}{32} & \frac{h^6}{64} & \frac{h^7}{128} & \frac{h^8}{256} \end{bmatrix} \tag{33}$$

Please cite this article in press as: Demasi L, ∞^6 Mixed plate theories based on the Generalized Unified Formulation. ..., Compos Struct (2008), doi:10.1016/j.compstruct.2008.07.011

Now consider the first layer where $k = 1$. The coordinates of the bottom and top surfaces of the first layer are:

$$z_{\text{bot}_{k=1}} = -\frac{h}{2} \quad z_{\text{top}_{k=1}} = \frac{h}{14} \tag{34}$$

For brevity only the matrices $\mathbf{K}^{k=1}_{u_x u_y}$, $\mathbf{K}^{k=1}_{u_x s_z}$ and $\mathbf{K}^{k=1}_{s_z s_z}$ are presented in Appendix B. To complete this example, consider a numerical case. Suppose (numbers chosen only to create the numerical example):

$$m = 2; \quad n = 3; \quad a = 10; \quad b = 15 \quad h = 7 \tag{35}$$

Assume that the materials are the following:

$$\text{Top layer} \begin{cases} E^{k=2}_{11} = 25 \quad E^{k=2}_{22} = 4 \quad E^{k=2}_{33} = 3 \\ G^{k=2}_{12} = \frac{1}{2} \quad G^{k=2}_{13} = \frac{3}{5} \quad G^{k=2}_{23} = \frac{1}{5} \\ \upsilon^{k=2}_{12} = \frac{1}{4} \quad \upsilon^{k=2}_{13} = \frac{27}{100} \quad \upsilon^{k=2}_{23} = \frac{29}{100} \quad h_{k=2} = \frac{3}{7}h \end{cases} \tag{36}$$

$$\text{Bottom layer} \begin{cases} E^{k=1}_{11} = 20 \quad E^{k=1}_{22} = 5 \quad E^{k=1}_{33} = 4 \\ G^{k=1}_{12} = \frac{1}{2} \quad G^{k=1}_{13} = \frac{3}{5} \quad G^{k=1}_{23} = \frac{1}{5} \\ \upsilon^{k=1}_{12} = \frac{1}{4} \quad \upsilon^{k=1}_{13} = \frac{27}{100} \quad \upsilon^{k=1}_{23} = \frac{29}{100} \quad h_{k=1} = \frac{4}{7}h \end{cases} \tag{37}$$

$\vartheta = 0$ is assumed for both layers. The numerical values for the matrices $\mathbf{K}^{k=1}_{u_x u_y}$, $\mathbf{K}^{k=1}_{u_x s_z}$, $\mathbf{K}^{k=1}_{s_z s_z}$, $\mathbf{K}^{k=2}_{u_x u_y}$, $\mathbf{K}^{k=2}_{u_x s_z}$ and $\mathbf{K}^{k=2}_{s_z s_z}$ are reported in Appendix C.

4. Conclusion

For the first time in the literature, the extension of the Generalized Unified Formulation to the case of mixed variational statements (in particular Reissner's Mixed Variational Theorem) and higher order shear deformation theories is presented.

The displacements, which have an equivalent single layer description, are expanded along the thickness by using a Taylor series. The stresses σ_{zx}, σ_{zy} and σ_{zz}, which have a layerwise description, are expanded along the thickness of each layer by using Legendre polynomials. Each variable can be treated separately from the others. This allows the writing, with a single formal derivation and software, of ∞^6 mixed higher order shear deformation theories. If the stresses are eliminated by using the Static Condensation Technique the resulting theory is formally identical to a "classical" displacement-based higher order shear deformation theory. If the stresses are not eliminated then a quasi-layerwise model is obtained. The new methodology, based on the use of the Generalized Unified Formulation, allows the user to freely change the orders of the expansion of the unknowns and experiment the best combination that better approximates the structural problem under investigation. The compatibility of the displacements and the equilibrium between two adjacent layers are enforced a priori.

All of the theories are generated by expanding 1×1 matrices (the *kernels* of the Generalized Unified Formulation), which are invariant with respect to the theory. Thus, with only 13 matrices (the kernels) ∞^6 theories can be generated without difficulties. These kernels are the same as the ones used in the layerwise case discussed in *Part II*.

The numerical performances and properties of mixed higher order theories will be discussed in *Part V* (see [22]) of the present work. It will be shown that the relative orders used for the displacements and stresses play a key role in the numerical stability of the solution.

Acknowledgement

The author thanks his sister Demasi Paola who inspired him with her strong will.

Appendix A. Explicit form of the matrices for a particular case

This appendix shows some of the matrices for the *top layer* ($k = 2$) of the structure described in Section 3. Theory EMC^{546}_{324} is considered.

$$\mathbf{K}^{k=2}_{u_x u_x} = \frac{\pi^2\left(C^{k=2}_{11} b^2 m^2 + C^{k=2}_{66} a^2 n^2\right)}{a^2 b^2} \begin{bmatrix} \frac{3}{7}h & \frac{6}{49}h^2 & \frac{57}{1372}h^3 & \frac{75}{4802}h^4 \\ \frac{6}{49}h^2 & \frac{57}{1372}h^3 & \frac{75}{4802}h^4 & \frac{8403}{1344560}h^5 \\ \frac{57}{1372}h^3 & \frac{75}{4802}h^4 & \frac{8403}{1344560}h^5 & \frac{2451}{941192}h^6 \\ \frac{75}{4802}h^4 & \frac{8403}{1344560}h^5 & \frac{2451}{941192}h^6 & \frac{411771}{368947264}h^7 \end{bmatrix} \tag{38}$$

$$\mathbf{K}^{k=2}_{u_x u_y} = \frac{\left(C^{k=2}_{12} + C^{k=2}_{66}\right) mn\pi^2}{ab} \begin{bmatrix} \frac{3}{7}h & \frac{6}{49}h^2 & \frac{57}{1372}h^3 \\ \frac{6}{49}h^2 & \frac{57}{1372}h^3 & \frac{75}{4802}h^4 \\ \frac{57}{1372}h^3 & \frac{75}{4802}h^4 & \frac{8403}{1344560}h^5 \\ \frac{75}{4802}h^4 & \frac{8403}{1344560}h^5 & \frac{2451}{941192}h^6 \end{bmatrix} \tag{39}$$

$$\mathbf{K}^{k=2}_{u_x s_x} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ +\frac{3}{14}h & -\frac{3}{7}h & 0 & 0 & 0 & \frac{3}{14}h \\ +\frac{15}{98}h^2 & -\frac{12}{49}h^2 & -\frac{3}{49}h^2 & 0 & 0 & \frac{9}{98}h^2 \\ +\frac{243}{2744}h^3 & -\frac{801}{6860}h^3 & -\frac{18}{343}h^3 & -\frac{27}{3430}h^3 & 0 & \frac{99}{2744}h^3 \end{bmatrix} \tag{40}$$

$$\mathbf{K}^{k=2}_{u_x s_z} = \frac{C^{k=2}_{13} m\pi}{a} \begin{bmatrix} -\frac{3}{14}h & \frac{3}{7}h & 0 & 0 & 0 & 0 & -\frac{3}{14}h \\ -\frac{15}{196}h^2 & \frac{6}{49}h^2 & \frac{3}{98}h^2 & 0 & 0 & 0 & -\frac{9}{196}h^2 \\ -\frac{81}{2744}h^3 & \frac{267}{6860}h^3 & \frac{6}{343}h^3 & \frac{9}{3430}h^3 & 0 & 0 & -\frac{33}{2744}h^3 \\ -\frac{2301}{192080}h^4 & \frac{321}{24010}h^4 & \frac{1089}{134456}h^4 & \frac{27}{12005}h^4 & \frac{81}{336140}h^4 & 0 & -\frac{699}{192080}h^4 \end{bmatrix} \tag{41}$$

$$\mathbf{K}^{k=2}_{u_y u_y} = \frac{\left(C^{k=2}_{66} b^2 m^2 + C^{k=2}_{22} a^2 n^2\right)\pi^2}{a^2 b^2} \begin{bmatrix} \frac{3}{7}h & \frac{6}{49}h^2 & \frac{57}{1372}h^3 \\ \frac{6}{49}h^2 & \frac{57}{1372}h^3 & \frac{75}{4802}h^4 \\ \frac{57}{1372}h^3 & \frac{75}{4802}h^4 & \frac{8403}{1344560}h^5 \end{bmatrix} \tag{42}$$

$$\mathbf{K}^{k=2}_{u_y s_y} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{3}{14}h & -\frac{3}{7}h & 0 & 0 & \frac{3}{14}h \\ \frac{15}{98}h^2 & -\frac{12}{49}h^2 & -\frac{3}{49}h^2 & 0 & \frac{9}{98}h^2 \end{bmatrix} \tag{43}$$

$$\mathbf{K}^{k=2}_{u_y s_z} = \frac{C^{k=2}_{23} n\pi}{b} \begin{bmatrix} -\frac{3}{14}h & \frac{3}{7}h & 0 & 0 & 0 & 0 & -\frac{3}{14}h \\ -\frac{15}{196}h^2 & \frac{6}{49}h^2 & \frac{3}{98}h^2 & 0 & 0 & 0 & -\frac{9}{196}h^2 \\ -\frac{81}{2744}h^3 & \frac{267}{6860}h^3 & \frac{6}{343}h^3 & \frac{9}{3430}h^3 & 0 & 0 & -\frac{33}{2744}h^3 \end{bmatrix} \tag{44}$$

$$\mathbf{K}^{k=2}_{u_z s_x} = \frac{m\pi}{a} \begin{bmatrix} \frac{3}{14}h & -\frac{3}{7}h & 0 & 0 & 0 & \frac{3}{14}h \\ \frac{15}{196}h^2 & -\frac{6}{49}h^2 & -\frac{3}{98}h^2 & 0 & 0 & \frac{9}{196}h^2 \\ \frac{81}{2744}h^3 & -\frac{267}{6860}h^3 & -\frac{6}{343}h^3 & -\frac{9}{3430}h^3 & 0 & \frac{33}{2744}h^3 \\ \frac{2301}{192080}h^4 & -\frac{321}{24010}h^4 & -\frac{1089}{134456}h^4 & -\frac{27}{12005}h^4 & -\frac{81}{336140}h^4 & \frac{699}{192080}h^4 \\ \frac{13539}{2689120}h^5 & -\frac{45753}{9411920}h^5 & -\frac{417}{117649}h^5 & -\frac{459}{336140}h^5 & -\frac{162}{588245}h^5 & \frac{3267}{2689120}h^5 \end{bmatrix} \tag{45}$$

$$\mathbf{K}^{k=2}_{u_z s_y} = \frac{n\pi}{b} \begin{bmatrix} \frac{3}{14}h & -\frac{3}{7}h & 0 & 0 & \frac{3}{14}h \\ \frac{15}{196}h^2 & -\frac{6}{49}h^2 & -\frac{3}{98}h^2 & 0 & \frac{9}{196}h^2 \\ \frac{81}{2744}h^3 & -\frac{267}{6860}h^3 & -\frac{6}{343}h^3 & -\frac{9}{3430}h^3 & \frac{33}{2744}h^3 \\ \frac{2301}{192080}h^4 & -\frac{321}{24010}h^4 & -\frac{1089}{134456}h^4 & -\frac{27}{12005}h^4 & \frac{699}{192080}h^4 \\ \frac{13539}{2689120}h^5 & -\frac{45753}{9411920}h^5 & -\frac{417}{117649}h^5 & -\frac{459}{336140}h^5 & \frac{3267}{2689120}h^5 \end{bmatrix} \tag{46}$$

$$\mathbf{K}^{k=2}_{u_z s_z} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{14}h & -\frac{3}{7}h & 0 & 0 & 0 & 0 & \frac{3}{14}h \\ \frac{15}{98}h^2 & -\frac{12}{49}h^2 & -\frac{3}{49}h^2 & 0 & 0 & 0 & \frac{9}{98}h^2 \\ \frac{243}{2744}h^3 & -\frac{801}{6860}h^3 & -\frac{18}{343}h^3 & -\frac{27}{3430}h^3 & 0 & 0 & \frac{99}{2744}h^3 \\ \frac{2301}{48020}h^4 & -\frac{642}{12005}h^4 & -\frac{1089}{33614}h^4 & -\frac{108}{12005}h^4 & -\frac{81}{84035}h^4 & 0 & \frac{699}{48020}h^4 \end{bmatrix} \quad (47)$$

$$\mathbf{K}^{k=2}_{s_x s_x} = C^{k=2}_{55} h \begin{bmatrix} -\frac{1}{7} & +\frac{3}{14} & +\frac{1}{14} & 0 & 0 & -\frac{1}{14} \\ +\frac{3}{14} & -\frac{18}{35} & 0 & +\frac{3}{35} & 0 & +\frac{3}{14} \\ +\frac{1}{14} & 0 & -\frac{10}{49} & 0 & +\frac{3}{49} & -\frac{1}{14} \\ 0 & +\frac{3}{35} & 0 & -\frac{2}{15} & 0 & 0 \\ 0 & 0 & +\frac{3}{49} & 0 & -\frac{54}{539} & 0 \\ -\frac{1}{14} & +\frac{3}{14} & -\frac{1}{14} & 0 & 0 & -\frac{1}{7} \end{bmatrix} \quad (48)$$

$$\mathbf{K}^{k=2}_{s_y s_y} = C^{k=2}_{44} h \begin{bmatrix} -\frac{1}{7} & +\frac{3}{14} & +\frac{1}{14} & 0 & -\frac{1}{14} \\ +\frac{3}{14} & -\frac{18}{35} & 0 & +\frac{3}{35} & +\frac{3}{14} \\ +\frac{1}{14} & 0 & -\frac{10}{49} & 0 & -\frac{1}{14} \\ 0 & +\frac{3}{35} & 0 & -\frac{2}{15} & 0 \\ -\frac{1}{14} & +\frac{3}{14} & -\frac{1}{14} & 0 & -\frac{1}{7} \end{bmatrix} \quad (49)$$

$$\mathbf{K}^{k=2}_{s_z s_z} = C^{k=2}_{33} h \begin{bmatrix} -\frac{1}{7} & +\frac{3}{14} & +\frac{1}{14} & 0 & 0 & 0 & -\frac{1}{14} \\ +\frac{3}{14} & -\frac{18}{35} & 0 & +\frac{3}{35} & 0 & 0 & +\frac{3}{14} \\ +\frac{1}{14} & 0 & -\frac{10}{49} & 0 & +\frac{3}{49} & 0 & -\frac{1}{14} \\ 0 & +\frac{3}{35} & 0 & -\frac{2}{15} & 0 & +\frac{1}{21} & 0 \\ 0 & 0 & +\frac{3}{49} & 0 & -\frac{54}{539} & 0 & 0 \\ 0 & 0 & 0 & +\frac{1}{21} & 0 & -\frac{22}{273} & 0 \\ -\frac{1}{14} & +\frac{3}{14} & -\frac{1}{14} & 0 & 0 & 0 & -\frac{1}{7} \end{bmatrix} \quad (50)$$

Appendix B. Explicit form of the matrices for a particular case

This appendix shows some of the matrices for the *bottom layer* ($k = 1$) of the structure described in Section 3. Theory EMC^{546}_{324} is considered.

$$\mathbf{K}^{k=1}_{u_x u_y} = \frac{(C^{k=1}_{12} + C^{k=1}_{66}) m n \pi^2}{ab} \begin{bmatrix} +\frac{4}{7}h & -\frac{6}{49}h^2 & +\frac{43}{1029}h^3 \\ -\frac{6}{49}h^2 & +\frac{43}{1029}h^3 & -\frac{75}{4802}h^4 \\ +\frac{43}{1029}h^3 & -\frac{75}{4802}h^4 & +\frac{2101}{336140}h^5 \\ -\frac{75}{4802}h^4 & +\frac{2101}{336140}h^5 & -\frac{2451}{941192}h^6 \end{bmatrix} \quad (51)$$

$$\mathbf{K}^{k=1}_{u_x s_z} = \frac{C^{k=1}_{13} m\pi}{a} \begin{bmatrix} -\frac{2}{7}h & +\frac{4}{7}h & 0 & 0 & 0 & 0 & -\frac{2}{7}h \\ +\frac{5}{147}h^2 & -\frac{6}{49}h^2 & +\frac{8}{147}h^2 & 0 & 0 & 0 & +\frac{13}{147}h^2 \\ -\frac{19}{2058}h^3 & +\frac{61}{1715}h^3 & -\frac{8}{343}h^3 & +\frac{32}{5145}h^3 & 0 & 0 & -\frac{67}{2058}h^3 \\ +\frac{131}{48020}h^4 & -\frac{279}{24010}h^4 & +\frac{158}{16807}h^4 & -\frac{48}{12005}h^4 & \frac{64}{84035}h^4 & 0 & +\frac{619}{48020}h^4 \end{bmatrix} \quad (52)$$

$$\mathbf{K}^{k=1}_{s_z s_z} = C^{k=1}_{33} h \begin{bmatrix} -\frac{4}{21} & +\frac{2}{7} & +\frac{2}{21} & 0 & 0 & 0 & -\frac{2}{21} \\ +\frac{2}{7} & -\frac{24}{35} & 0 & +\frac{4}{35} & 0 & 0 & +\frac{2}{7} \\ +\frac{2}{21} & 0 & -\frac{40}{147} & 0 & +\frac{4}{49} & 0 & -\frac{2}{21} \\ 0 & +\frac{4}{35} & 0 & -\frac{8}{45} & 0 & +\frac{4}{63} & 0 \\ 0 & 0 & +\frac{4}{49} & 0 & -\frac{72}{539} & 0 & 0 \\ 0 & 0 & 0 & +\frac{4}{63} & 0 & -\frac{88}{819} & 0 \\ -\frac{2}{21} & +\frac{2}{7} & -\frac{2}{21} & 0 & 0 & 0 & -\frac{4}{21} \end{bmatrix} \quad (53)$$

Appendix C. Numerical expressions for some matrices

This appendix shows some of the matrices for the *top layer* ($k = 2$) and *bottom layer* of the structure described in Section 3. Theory EMC^{546}_{324} is considered.

$$\mathbf{K}^{k=1}_{u_x u_y} = \begin{bmatrix} 2.79 & -4.19 & 10.01 \\ -4.19 & 10.01 & -26.20 \\ 10.01 & -26.20 & 73.40 \\ -26.20 & 73.40 & -214.07 \end{bmatrix} \quad (54)$$

$$\mathbf{K}^{k=1}_{u_x s_z} = \begin{bmatrix} -0.44 & 0.87 & 0 & 0 & 0 & 0 & -0.44 \\ 0.36 & -1.31 & 0.58 & 0 & 0 & 0 & 0.95 \\ -0.69 & 2.67 & -1.75 & 0.47 & 0 & 0 & -2.44 \\ 1.43 & -6.10 & 4.93 & -2.10 & 0.40 & 0 & 6.77 \end{bmatrix} \quad (55)$$

$$\mathbf{K}^{k=1}_{s_z s_z} = \begin{bmatrix} -0.30 & 0.45 & 0.15 & 0 & 0 & 0 & -0.15 \\ 0.45 & -1.09 & 0 & 0.18 & 0 & 0 & 0.45 \\ 0.15 & 0 & -0.43 & 0 & 0.13 & 0 & -0.15 \\ 0 & 0.18 & 0 & -0.28 & 0 & 0.10 & 0 \\ 0 & 0 & 0.13 & 0 & -0.21 & 0 & 0 \\ 0 & 0 & 0 & 0.10 & 0 & -0.17 & 0 \\ -0.15 & 0.45 & -0.15 & 0 & 0 & 0 & -0.30 \end{bmatrix} \quad (56)$$

$$\mathbf{K}^{k=2}_{u_x u_y} = \begin{bmatrix} 1.79 & 3.58 & 8.50 \\ 3.58 & 8.50 & 22.36 \\ 8.50 & 22.36 & 62.62 \\ 22.36 & 62.62 & 182.65 \end{bmatrix} \quad (57)$$

$$\mathbf{K}^{k=2}_{u_x s_z} = \begin{bmatrix} -0.33 & 0.65 & 0 & 0 & 0 & 0 & -0.33 \\ -0.82 & 1.30 & 0.33 & 0 & 0 & 0 & -0.49 \\ -2.20 & 2.90 & 1.30 & 0.20 & 0 & 0 & -0.90 \\ -6.25 & 6.98 & 4.23 & 1.17 & 0.13 & 0 & -1.90 \end{bmatrix} \quad (58)$$

$$\mathbf{K}^{k=2}_{s_z s_z} = \begin{bmatrix} -0.31 & 0.46 & 0.15 & 0 & 0 & 0 & -0.15 \\ 0.46 & -1.11 & 0 & 0.18 & 0 & 0 & 0.46 \\ 0.15 & 0 & -0.44 & 0 & 0.13 & 0 & -0.15 \\ 0 & 0.18 & 0 & -0.29 & 0 & 0.10 & 0 \\ 0 & 0 & 0.13 & 0 & -0.22 & 0 & 0 \\ 0 & 0 & 0 & 0.10 & 0 & -0.17 & 0 \\ -0.15 & 0.46 & -0.15 & 0 & 0 & 0 & -0.31 \end{bmatrix} \quad (59)$$

References

[1] Cauchy AL. Sure l'equilibre et le movement d'une plaque solide. Excercies Matematique 1828;3:328–55.
[2] Poisson SD. Memoire sur l'equilibre et le mouvement des corps elastique. Mem l'Acad Sci 1829;8:357.
[3] Kirchhoff G. Über das Gleichgewicht und die Bewegung einer elastishen Scheibe. J Angew Math 1850;40:51–88.
[4] Reddy JN. Mechanics of laminated composite plates, theory and analysis. 2nd ed. Boca Raton, London, New York, Washington, DC: CRC Press; 2004.
[5] Reissner E. The effect of transverse shear deformation on the bending of elastic plates. J Appl Mech 1945;12:69–76.
[6] Mindlin RD. Influence of rotatory inertia and shear in flexural motion of isotropic elastic plates. J Appl Mech 1951;18:1031–6.
[7] Kärger L, Wetzel A, Rolfes R, Rohwer K. A three-layered sandwich element with improved transverse shear stiffness and stress based on FSDT. Comput Struct 2007. doi:10.1016/j.compstruc.2006.02.007.

[8] Swaminathan K, Patil SS. Analytical solutions using higher order refined computational model with 12 degrees of freedom for the free vibration analysis of antisymmetric angle-ply plates. Compos Struct 2007;1:1–2. doi:10.1016/j.compstruct.2007.01.001.
[9] Batra RC, Vidoli S. Higher order piezoelectric plate theory derived from a three-dimensional variational principle. AIAA J 2002;40(1):91–104.
[10] Kant T, Swaminathan K. Free vibration of isotropic, orthotropic, and multilayer plates based on higher order refined theories. J Sound Vibrat 2001;241(2): 319–27.
[11] Wu Zhen, Cheung YK, Lo SH, Wanji Chen. Effects of higher-order global-local shear deformations on bending, vibration and buckling of multilayered plates. Compos Struct 2007;1:1–2. doi:10.1016/j.compstruct.2007.01.017.
[12] Carrera E. Theories and finite elements for multilayered plates and shells: a unified compact formulation with numerical assessment and benchmarking. Arch Comput Methods Eng 2003;10(3):215–96.
[13] Carrera E. Historical review of zig-zag theories for multilayered plates and shells. Appl Mech Rev 2003;56(3).
[14] Demasi L. ∞^6 Mixed plate theories based on the Generalized Unified Formulation. Part IV: zig-zag theories. Compos Struct 2008. doi:10.1016/j.compstruct.2008.07.010.
[15] Demasi L. ∞^3 Hierarchy plate theories for thick and thin composite plates, Compos Struct; 2007, doi:10.1016/jcompstruct.2007.08.004 (available on-line since August 22nd 2007).
[16] Demasi L. Three-dimensional closed form solutions and ∞^3 theories for orthotropic plates. Mech Adv Mater Struct, submitted for publication.
[17] Reissner E. On a certain mixed variational theory and a proposed application. Int J Numer Methods Eng 1984;20:1366–8.
[18] Reissner E. On a mixed variational theorem and on shear deformable plate theory. Int J Numer Methods Eng 1986;23:193–8.
[19] Demasi L. ∞^6 Mixed plate theories based on the Generalized Unified Formulation. Part I: governing equations. Compos Struct 2008. doi:10.1016/j.compstruct.2008.07.013.
[20] Demasi L. ∞^6 Mixed plate theories based on the Generalized Unified Formulation. Part II: layerwise theories. Compos Struct 2008. doi:10.1016/j.compstruct.2008.07.012.
[21] Kreyszig E. Advanced engineering mathematics. John Wiley & Sons, INC; 1999.
[22] Demasi L. ∞^6 Mixed plate theories based on the Generalized Unified Formulation. Part V: results. Compos Struct 2008. doi:10.1016/j.compstruct.2008.07.009.

Please cite this article in press as: Demasi L, ∞^6 Mixed plate theories based on the Generalized Unified Formulation. ..., Compos Struct (2008), doi:10.1016/j.compstruct.2008.07.011