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∞^6 mixed plate theories based on the Generalized Unified Formulation. Part IV: zig-zag theories

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ABSTRACT

The generalized unified formulation was introduced in *Part I* for the case of plate theories based upon Reissner's mixed variational theorem. *Part II* analyzed the case of layerwise theories and *Part III* studied advanced mixed higher order shear deformation theories.

In this work the generalized unified formulation is applied, for the first time in the literature, to the case of advanced mixed higher order zig-zag theories. The so called zig-zag form of the displacements is enforced a priori by the adoption of Murakami's zig-zag function. An equivalent single layer description of the displacements u_x, u_y and u_z is adopted. The out-of-plane stresses σ_{zx}, σ_{zy} and σ_{zz} have a layerwise description. The compatibility of the displacements and the equilibrium of the transverse stresses between two adjacent layers are enforced a *priori*. ∞^6 mixed higher order zig-zag theories are therefore presented. The kernels have the same formal expressions as the ones used in the layerwise theories analyzed in *Part II* and in the higher order shear deformation theories presented in *Part III*.

1. Introduction

1.1. Zig-zag theories: main concepts

As explained in *Part III* (Ref. [1]), classical theories [2–4] may not be sufficient to capture the behavior of relatively thick plates or multilayered composite structures with strong transverse anisotropy. Therefore, several models were developed by many authors. First order shear deformation theories [5–8] improve the axiomatic models by allowing non-zero shear strains γ_{zx} and γ_{zy}. Refinements of these models can be obtained by increasing the orders of the expansion of the different variables [9–12]. However, even these improvements are not effective in complex cases with localized loads or high transverse anisotropy. More accurate models prescribe a displacement field (and stress field in the case of mixed approaches) within each layer. We have then the so called layerwise theories [13–21], which are very accurate especially in the cases in which the continuity of the out-of-plane stresses is enforced a priori [17–19]. But layerwise theories have the drawback of being computationally expensive. Is there something more accurate than the higher order theories with a similar number of degrees of freedoms? The answer is yes, *zig-zag* theories (see [22] for an overview). The concept behind zig-zag theories and zig-zag form of the displacements is the following. The equilibrium between two adjacent layers implies that the out-of-plane stresses are equal at the interface. These stresses can be thought as a combination of strains multiplied by some coefficients that depend on the material of each layer (Hooke's law). In general, two layers have different mechanical properties and, therefore, different strains are required to obtain equilibrium. The strains are related to the derivatives of the displacements (geometric relations). Thus, different strains imply different slopes of the displacements. This fact leads to the zig-zag form of the displacements (see Fig. 1).

A very large amount of literature has been devoted to the formulation of axiomatic zig-zag theories that take into account these requirements. Following the terminology introduced in [22], three different *categories* of zig-zag theories can be created. In particular:

- Lekhnitskii multilayered theory (LMT).
- Ambartsumian multilayered theory (AMT).
- Reissner multilayered theory (RMT).

LMT was introduced for the particular case of cantilevered multilayered beam (Ref. [23]) and almost ignored in subsequent works with a few exception (see Ren's works in Refs. [24,25]). A summary of the main facts of LMT is presented in Ref. [22]. Ambartsumian work ([26–29]) was an extension of Reissner–Mindlin theory (Refs. [6,7]). RMT is based on Reissner's mixed variational theorem (RMVT) (see [30,31]). The contribution of the present work is in the framework of RMT; thus, RMT will be discussed in more detail. Following the "historical" review (see [22]), the first application of Reissner's mixed variational theorem was made by Murakami [32], who introduced the zig-zag function (MZZF). MZZF has the

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advantage of being simple and reproducing the discontinuity of the first derivative of the displacements in the thickness direction. Extensive work based on RMVT and MZZF was presented by Carrera and co-authors (see for example [33–35]). It has also been demonstrated that the usage of MZZF is more effective than increasing the orders used in the expansion of the variables (see Refs. [36,37] for FEM applications). It will be demonstrated in *Part V* (Ref. [38]) that this is only partially true: when the orders of the displacements and stresses are varied some combinations of the orders may not converge to the correct solution.

1.2. What are the new contributions of this work

The generalized unified formulation (GUF) is a generalization[1] of Carrera's unified formulation (see [39,40]) and extended to the case of RMVT-based theories in *Part I* (Ref. [41]). *Parts II* and *III* (Refs. [42,1]) showed that 13 kernels (1×1 matrices) could generate an infinite number of layerwise theories and mixed higher order shear deformation theories. *Part IV* will show that *the same* kernels are also valid for the generation of infinite zig-zag theories with orders of the displacements *and* out-of-plane stresses independently postulated. An extensive adoption of MZZF will be shown very effective in the generation of the zig-zag theories.

2. Theoretical derivation of ∞^6 advanced mixed higher order zig-zag theories

A theory, in which the in-plane displacements are expanded along the thickness by using a cubic polynomial and the out-of-plane displacement u_z is parabolic, is considered

$$\text{Theory I}: \begin{cases} u_x = u_{x_0} + z\phi_{1_{u_x}} + z^2\phi_{2_{u_x}} + z^3\phi_{3_{u_x}} \\ u_y = u_{y_0} + z\phi_{1_{u_y}} + z^2\phi_{2_{u_y}} + z^3\phi_{3_{u_y}} \\ u_z = u_{z_0} + z\phi_{1_{u_z}} + z^2\phi_{2_{u_z}} \end{cases} \quad (1)$$

In general the layers present different mechanical characteristics. Therefore, Hooke's coefficients relative to layer k are different than the coefficients valid for layer $k+1$. The equilibrium between two adjacent layers is satisfied if the out-of-plane stresses are continuous functions along the thickness. But the stresses are related to the derivatives of the displacements (Hooke's law). So, continuity of the out-of-plane stresses implies discontinuity (in general) of the first derivatives of the displacements. This effect can be named "zig-zag form of the displacements". Murakami suggested to take into account this fact by introducing a zig-zag function (Murakami's zig-zag function). This concept is shown in Fig. 1.

Theory I (see Eq. (1)) is then "improved" as follows (Theory II is the resulting theory):

$$\text{Theory II}: \begin{cases} u_x = u_{x_0} + z\phi_{1_{u_x}} + z^2\phi_{2_{u_x}} + z^3\phi_{3_{u_x}} + \overbrace{(-1)^k\zeta_k u_{x_Z}}^{\text{Extra term for Zig-Zag effect}} \\ u_y = u_{y_0} + z\phi_{1_{u_y}} + z^2\phi_{2_{u_y}} + z^3\phi_{3_{u_y}} + \overbrace{(-1)^k\zeta_k u_{y_Z}}^{\text{Extra term for Zig-Zag effect}} \\ u_z = u_{z_0} + z\phi_{1_{u_z}} + z^2\phi_{2_{u_z}} + \overbrace{(-1)^k\zeta_k u_{z_Z}}^{\text{Extra term for Zig-Zag effect}} \end{cases} \quad (2)$$

When a particular layer k is considered then z is contained in the interval $[z_{\text{bot}_k}, z_{\text{top}_k}]$. That is,

$$z_{\text{bot}_k} \leqslant z \leqslant z_{\text{top}_k} \quad (3)$$

[1] The definition of "Carrera's unified formulation" is a terminology introduced by the author in Ref. [40] to uniquely identify a compact notation and formalism introduced by Carrera for the 2D modelization of multilayered plates.

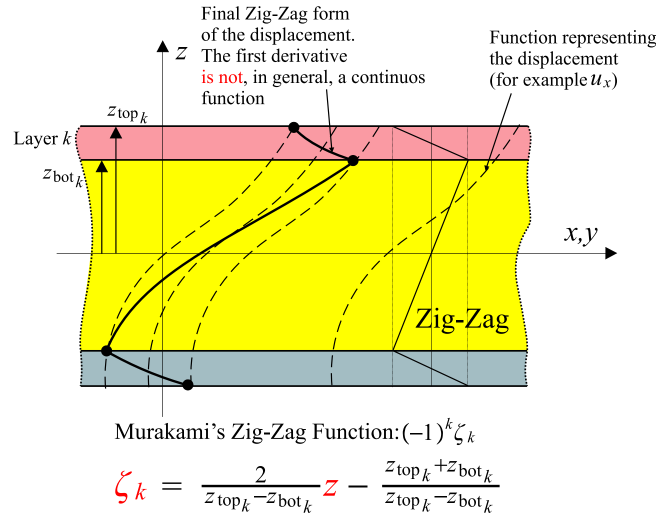

Fig. 1. Zig-zag form of the displacements and Murakami's zig-zag function.

z_{bot_k} is the thickness coordinate of the *bottom surface of layer k* and z_{top_k} is the thickness coordinate of the *top surface of layer k*. The origin of the coordinate system is in the middle plane of the plate (see Fig. 1).

The quantity ζ_k is the non-dimensional thickness coordinate of layer k and is included in the interval $[-1, 1]$. The following expression is verified:

$$\zeta_k = \frac{2}{z_{\text{top}_k} - z_{\text{bot}_k}} z - \frac{z_{\text{top}_k} + z_{\text{bot}_k}}{z_{\text{top}_k} - z_{\text{bot}_k}} \quad (4)$$

z is measured from the middle plane of the plate. In Eq. (2), valid for a theory with zig-zag form of the displacements included, the following can be observed:

- In Murakami's zig-zag functions the term $(-1)^k$ is present. k is the integer representing the ID of a generic layer, $k = 1$ is for the bottom layer and $k = N_l$ is for the top layer (N_l is the number of layers). The term $(-1)^k$ enforces the *discontinuity* of the first derivative (thickness direction) of the displacement. For example, in layer k the derivative with respect to z of the zig-zag term relative to the component u_x is

$$\frac{\mathrm{d}[(-1)^k\zeta_k u_{x_Z}]}{\mathrm{d}z} = (-1)^k u_{x_Z}\frac{\mathrm{d}\zeta_k}{\mathrm{d}z} = (-1)^k u_{x_Z}\frac{2}{z_{\text{top}_k} - z_{\text{bot}_k}} \quad (5)$$

  As can be seen the term $(-1)^k$ strongly affects the sign of the derivative.
- The displacements still have an equivalent single layer description. In fact, the terms u_{x_Z}, u_{y_Z} and u_{z_Z} are *independent* on the actual layer and defined for the whole plate.
- The zig-zag form of the displacements is taken into account *a priori* by adding only *three* degrees of freedom (u_{x_Z}, u_{y_Z} and u_{z_Z} respectively). This is a general property and does not depend on the orders used for the expansion of the different variables. That is, a generic theory can take into account the zig-zag form of the displacements by adding only three extra degrees of freedoms, as was done in Eqs. (1) and (2) for the case of Theory I.

For *each displacement component* the concepts of the generalized unified formulation (see *Part I*, Ref. [41]) can be applied. For example, the displacement u_x in Eq. (2) is written as

$$\begin{aligned} u_x &= u_{x_0} + z\phi_{1_{u_x}} + z^2\phi_{2_{u_x}} + z^3\phi_{3_{u_x}} + (-1)^k\zeta_k u_{x_Z} \\ &= {}^xF_t u_{x_t} + {}^xF_2 u_{x_2} + {}^xF_3 u_{x_3} + {}^xF_4 u_{x_4} + {}^xF_b u_{x_b} \end{aligned} \quad (6)$$

where

$$ {}^{x}F_t = 1;\quad {}^{x}F_2 = z;\quad {}^{x}F_3 = z^2;\quad {}^{x}F_4 = z^3;\quad {}^{x}F_b = (-1)^k \zeta_k $$
$$ u_{x_t} = u_{x_0};\quad u_{x_2} = \phi_{1_{u_x}};\quad u_{x_3} = \phi_{2_{u_x}};\quad u_{x_4} = \phi_{3_{u_x}};\quad u_{x_b} = u_{x_Z} \qquad (7) $$

The GUF for the displacement u_x is

$$ u_x = {}^{x}F_{\alpha_{u_x}} u_{x\alpha_{u_x}} \quad \alpha_{u_x} = t, l, b;\quad l = 2, \ldots, N_{u_x} + 1 \qquad (8) $$

where, in the example, $N_{u_x} = 3$. For the displacement u_y the formal procedure produces similar results

$$ u_y = {}^{y}F_{\alpha_{u_y}} u_{y\alpha_{u_y}} \quad \alpha_{u_y} = t, m, b;\quad m = 2, \ldots, N_{u_y} + 1 \qquad (9) $$

The displacement u_z is only parabolic, but the representation is formally the same

$$ u_z = {}^{z}F_{\alpha_{u_z}} u_{z\alpha_{u_z}} \quad \alpha_{u_z} = t, n, b;\quad n = 2, \ldots, N_{u_z} + 1 \qquad (10) $$

The following are observed:

- The superscript k is not present. In fact, in equivalent single layer models the displacement fields have a description at plate level and not at layerwise level.
- The generalized unified formulation (Eqs. (8)–(10)) is formally equivalent to the layerwise case (see *Part II*, Ref. [42]). In fact, the layerwise GUF *for the same number of terms* would be

$$ \begin{aligned} u_x^k &= {}^{x}F_{\alpha_{u_x}} u^k_{x\alpha_{u_x}} \quad \alpha_{u_x} = t, l, b;\quad l = 2, \ldots, N_{u_x} + 1 \\ u_y^k &= {}^{y}F_{\alpha_{u_y}} u^k_{y\alpha_{u_y}} \quad \alpha_{u_y} = t, m, b;\quad m = 2, \ldots, N_{u_y} + 1 \\ u_z^k &= {}^{z}F_{\alpha_{u_z}} u^k_{z\alpha_{u_z}} \quad \alpha_{u_z} = t, n, b;\quad n = 2, \ldots, N_{u_z} + 1 \end{aligned} \qquad (11) $$

  The similarity between the equivalent single layer case with zig-zag function and the layerwise case suggests that it is possible to use the layerwise generalized unified formulation for the zig-zag case too. that is, Eq. (11) can be used for the equivalent single layer case with zig-zag form of the displacements. The fact that the displacement field *does not* have a layerwise description (see, for example, Theory II, explicitly written in Eq. (2)) is taken into account when the *assembling in the thickness direction* of the layer matrices is considered.

Reissner's mixed variational theorem is adopted. Thus, the unknowns are the displacements and out-of-plane shear and normal stresses. The continuity of these stresses is enforced by using a *layerwise* description for them. The following terminologies are introduced here:

- *Reissner's mixed variational theorem-based zig-zag theories (RZZT)*
  These theories have an equivalent single layer description of the displacement fields and include the zig-zag form of the displacements (a typical example is reported in Eq. (2)). A layerwise description of the out-of-plane stresses is used. However, the stresses are eliminated by using the static condensation technique (SCT). Therefore, formally, the theories have a final equivalent single layer description. This approach gives, for example, a theory similar to Theory II (Eq. (2)) in which the stresses are not assumed.
- *Quasi-layerwise Reissner's mixed variational theorem-based zig-zag theories (QLRZZT)*
  These theories have an equivalent single layer description of the displacement fields and a layerwise description of the out-of-plane stresses. The zig-zag form of the displacements is enforced a priori by using MZZF, as in the case of RZZT. The SCT *is not* used. Thus, some quantities (the displacements) are described as in the ESL approach and some quantities (the stresses σ_{xz}, σ_{yz} and σ_{zz}) are described as layerwise quantities.

A Navier-type solution will be considered in this paper. Therefore, considering that the static condensation technique is applied at plate level, the difference between RZZT and QLRZZT is practically only formal. The generalized unified formulation, valid for both RZZT and QLRZZT, is then the following (the simplified notation for the stresses used is: $\sigma^k_{xz} = s^k_x, \sigma^k_{yz} = s^k_y, \sigma^k_{zz} = s^k_z$):

$$ \begin{aligned} u_x^k &= {}^{x}F_t u^k_{x_t} + {}^{x}F_l u^k_{x_l} + {}^{x}F_b u^k_{x_b} = {}^{x}F_{\alpha_{u_x}} u^k_{x\alpha_{u_x}} \\ &\quad \alpha_{u_x} = t, l, b; l = 2, \ldots, N_{u_x} + 1 \\ u_y^k &= {}^{y}F_t u^k_{y_t} + {}^{y}F_m u^k_{y_m} + {}^{y}F_b u^k_{y_b} = {}^{y}F_{\alpha_{u_y}} u^k_{y\alpha_{u_y}} \\ &\quad \alpha_{u_y} = t, m, b; m = 2, \ldots, N_{u_y} + 1 \\ u_z^k &= {}^{z}F_t u^k_{z_t} + {}^{z}F_n u^k_{z_n} + {}^{z}F_b u^k_{z_b} = {}^{z}F_{\alpha_{u_z}} u^k_{z\alpha_{u_z}} \\ &\quad \alpha_{u_z} = t, n, b; n = 2, \ldots, N_{u_z} + 1 \\ s_x^k &= {}^{x}\mathcal{F}_t s^k_{x_t} + {}^{x}\mathcal{F}_p s^k_{x_p} + {}^{x}\mathcal{F}_b s^k_{x_b} = {}^{x}\mathcal{F}_{\alpha_{s_x}} s^k_{x\alpha_{s_x}} \\ &\quad \alpha_{s_x} = t, p, b; p = 2, \ldots, N_{s_x} \\ s_y^k &= {}^{y}\mathcal{F}_t s^k_{y_t} + {}^{y}\mathcal{F}_q s^k_{y_q} + {}^{y}\mathcal{F}_b s^k_{y_b} = {}^{y}\mathcal{F}_{\alpha_{s_y}} s^k_{y\alpha_{s_y}} \\ &\quad \alpha_{s_y} = t, q, b; q = 2, \ldots, N_{s_y} \\ s_z^k &= {}^{z}\mathcal{F}_t s^k_{z_t} + {}^{z}\mathcal{F}_r s^k_{z_r} + {}^{z}\mathcal{F}_b s^k_{z_b} = {}^{z}\mathcal{F}_{\alpha_{s_z}} s^k_{z\alpha_{s_z}} \\ &\quad \alpha_{s_z} = t, r, b; r = 2, \ldots, N_{s_z} \end{aligned} \qquad (12) $$

The functions of the thickness coordinate ${}^{x}F_t, {}^{y}F_t, {}^{z}F_t, {}^{x}F_l, {}^{y}F_m, {}^{z}F_n$ are assumed to be of the type $1\ z\ z^2\ z^3 \ldots$ ${}^{x}F_b, {}^{y}F_b$ and ${}^{z}F_b$ contain the zig-zag functions. This choice is made in order to be "consistent" with the usual approach used in the literature for equivalent single layer models. However, the out-of-plane stresses are modeled as layerwise quantities. As done in *Parts II* and *III* (Ref. [42,1]) a combination of Legendre polynomials is used. In detail, the functions used in the expansions of the displacements are

$$ \begin{aligned} &{}^{x}F_t = 1 \quad {}^{y}F_t = 1 \quad {}^{z}F_t = 1 \\ &{}^{x}F_2 = z \quad {}^{y}F_2 = z \quad {}^{z}F_2 = z \\ &{}^{x}F_3 = z^2 \quad {}^{y}F_3 = z^2 \quad {}^{z}F_3 = z^2 \\ &\ldots \\ &{}^{x}F_l = z^{l-1} \quad {}^{y}F_m = z^{m-1} \quad {}^{z}F_n = z^{n-1} \\ &\ldots \\ &{}^{x}F_b = (-1)^k \zeta_k \quad {}^{y}F_b = (-1)^k \zeta_k \quad {}^{z}F_b = (-1)^k \zeta_k \end{aligned} \qquad (13) $$

The functions used in the expansion of the stresses $\sigma_{xz} = s^k_x, \sigma_{yz} = s^k_y$ and $\sigma_{zz} = s^k_z$ as for the case showed in *Part III*, are

$$ \begin{aligned} &{}^{x}\mathcal{F}_t = {}^{y}\mathcal{F}_t = {}^{z}\mathcal{F}_t = \frac{P_0 + P_1}{2}, \\ &{}^{x}\mathcal{F}_b = {}^{y}\mathcal{F}_b = {}^{z}\mathcal{F}_b = \frac{P_0 - P_1}{2} \\ &{}^{x}\mathcal{F}_p = P_p - P_{p-2}, \quad p = 2, 3, \ldots, N_{s_x} \\ &{}^{y}\mathcal{F}_q = P_q - P_{q-2}, \quad q = 2, 3, \ldots, N_{s_y} \\ &{}^{z}\mathcal{F}_r = P_r - P_{r-2}, \quad r = 2, 3, \ldots, N_{s_z} \end{aligned} \qquad (14) $$

where P_0, P_1, P_p, P_q, P_r are the Legendre polynomials of order $0, 1, p, q$ and r respectively. More details about these polynomials can be found in *Parts II* and *III*.

Reissner's mixed variational theorem-based zig-zag theories (RZZT), indicated with acronyms, are shown in Fig. 2. The quasi-layerwise Reissner's mixed variational theorem-based zig-zag theories (QLRZZT), indicated with acronyms, are shown in Fig. 3. In both RZZT and QLRZZT cases it is then possible to create a class of theories by changing the order used for the displacements and stresses. Suppose, for example, that a theory has the following data: $N_{u_x} = 3, N_{u_y} = 2, N_{u_z} = 4, N_{s_x} = 5, N_{s_y} = 4, N_{s_z} = 6$. The corresponding RZZT theory is indicated as EMZC_{324}^{546}. The first letter "E" means "equivalent single layer" theory, the second letter "M" means that a mixed variational statement is used (in the present case Reissner's variational theorem), the third letter "Z" means that the zig-zag form of the displacement is enforced a priori and the fourth letter "C" means that the continuity of the out-of-plane stresses is enforced (this operation is done in the assembling in the thickness direction). The subscripts are the orders of the polynomials used for the displacements. The superscripts are the orders of the Legendre polynomials used for the out-of-plane stresses. In general the acronym is then built as follows: $\text{EMZC}_{N_{u_x}N_{u_y}N_{u_z}}^{N_{s_x}N_{s_y}N_{s_z}}$.

Similarly, with the same orders used for displacements and stresses it is possible to build QLRZZT theories. These theories are indicated with the acronym $\text{QLMZC}_{N_{u_x}N_{u_y}N_{u_z}}^{N_{s_x}N_{s_y}N_{s_z}}$. "Q" stands for "quasi" and "L" stands for "layerwise". In this paper there is no numerical difference between a generic theory $\text{EMZC}_{N_{u_x}N_{u_y}N_{u_z}}^{N_{s_x}N_{s_y}N_{s_z}}$ and the corresponding theory $\text{QLMZC}_{N_{u_x}N_{u_y}N_{u_z}}^{N_{s_x}N_{s_y}N_{s_z}}$. The reason, as stated in *Parts II* and *III*, is that the static condensation technique is performed at plate level. But when FEM applications are considered this is no longer true, since it is convenient to perform the static condensation at element level.

2.1. Expansion of the matrices

The expansion of the matrices is an important part of the generation of one of the ∞^6 theories. This operation is done *at layer level* and does not present any particular differences with respect to the ESL theories without zig-zag terms (see *Part III*). The main difference is in the number of DOFs, which in the present case is increased by 3 to take into account the zig-zag form of the displacements u_x, u_y and u_z (see, for example, Eqs. (1) and (2)).

To explain how the expansion is performed, consider theory EMZC_{213}^{546} (theory QLMZC_{213}^{546} is equivalent at this point).

In theory EMZC_{213}^{546} the number of degrees of freedom (at layer level) is the following:

$$\begin{aligned}
[\text{DOF}]_{u_x}^k &= N_{u_x} + 2 = 2 + 2 = 4\\
[\text{DOF}]_{u_y}^k &= N_{u_y} + 2 = 1 + 2 = 3\\
[\text{DOF}]_{u_z}^k &= N_{u_z} + 2 = 3 + 2 = 5\\
[\text{DOF}]_{s_x}^k &= N_{s_x} + 1 = 5 + 1 = 6\\
[\text{DOF}]_{s_y}^k &= N_{s_y} + 1 = 4 + 1 = 5\\
[\text{DOF}]_{s_z}^k &= N_{s_z} + 1 = 6 + 1 = 7
\end{aligned} \tag{15}$$

It has to be pointed out that in the case of the corresponding ESL theory without zig-zag term (theory EMC_{213}^{546}, see *Part III*) we would have the following:

$$\begin{aligned}
[\text{DOF}]_{u_x}^k &= N_{u_x} + 1 = 2 + 1 = 3\\
[\text{DOF}]_{u_y}^k &= N_{u_y} + 1 = 1 + 1 = 2\\
[\text{DOF}]_{u_z}^k &= N_{u_z} + 1 = 3 + 1 = 4\\
[\text{DOF}]_{s_x}^k &= N_{s_x} + 1 = 5 + 1 = 6\\
[\text{DOF}]_{s_y}^k &= N_{s_y} + 1 = 4 + 1 = 5\\
[\text{DOF}]_{s_z}^k &= N_{s_z} + 1 = 6 + 1 = 7
\end{aligned} \tag{16}$$

From the number of degrees of freedom (see Eq. (15)) it is possible to calculate the size of the layer matrices. For example, when matrix $K_{u_x s_z}^{k\alpha_{u_x}\beta_{s_z}}$ is expanded then the final size at layer level is $[\text{DOF}]_{u_x}^k \times [\text{DOF}]_{s_z}^k$. In the example relative to theory EMZC_{213}^{546}, matrix $K_{u_x s_z}^{k\alpha_{u_x}\beta_{s_z}}$ at layer level (indicated as $\mathbf{K}_{u_x s_z}^k$) is obtained as explained in Fig. 4. It should be noticed that at this point there is no difference between the case of RZZT and QLRZZT. Fig. 4 is valid for both cases.

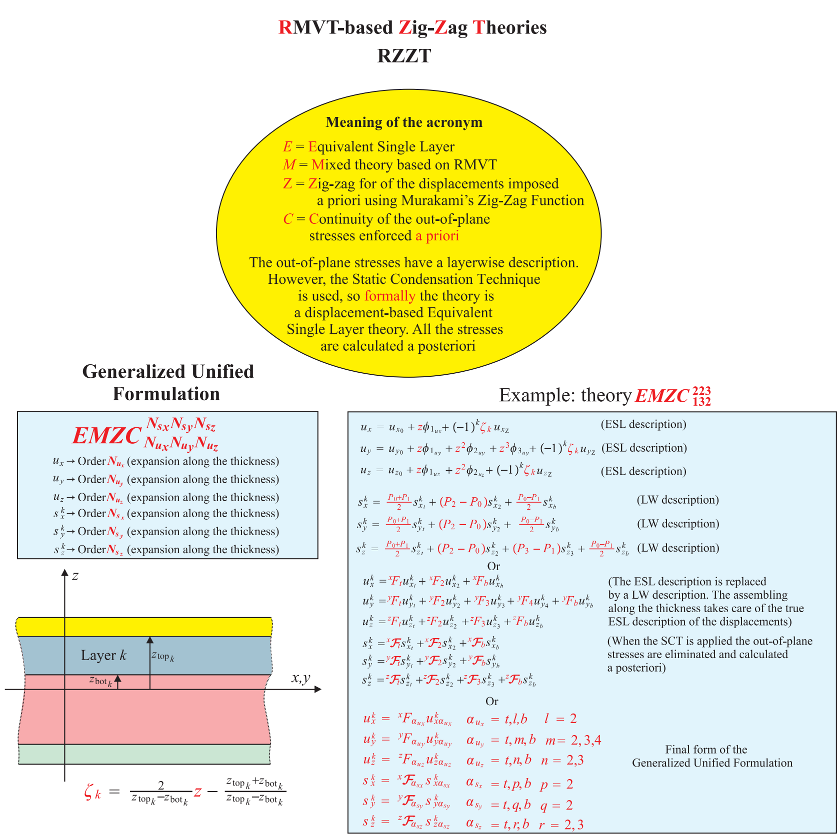

Fig. 2. Reissner's mixed variational theorem-based zig-zag theories: acronyms used.

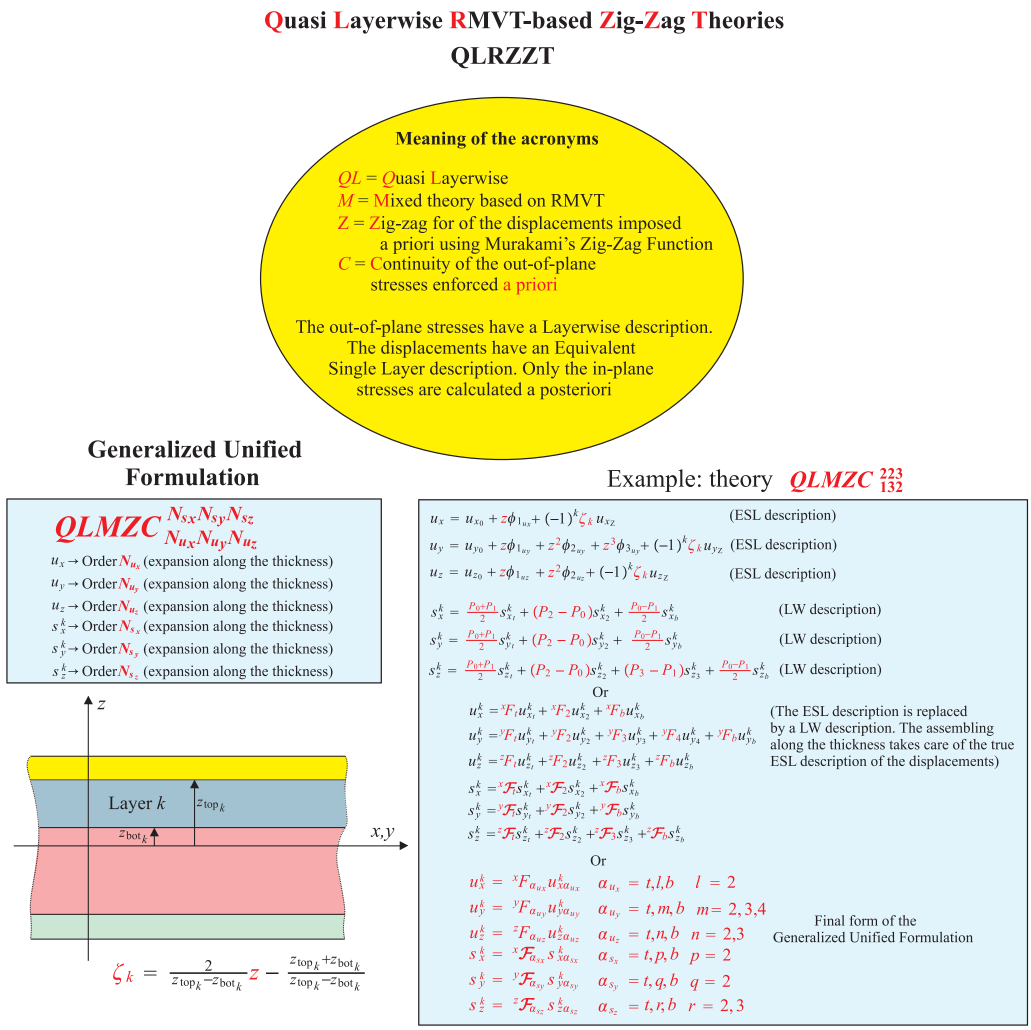

Fig. 3. Quasi-layerwise Reissner's mixed variational theorem-based zig-zag theories: acronyms used.

2.2. Assembling in the thickness direction

In addition to the compatibility of the displacements, the equilibrium between two adjacent layers implies that $s_{x_t}^k = s_{x_b}^{(k+1)}, s_{y_t}^k = s_{y_b}^{(k+1)}$ and $s_{z_t}^k = s_{z_b}^{(k+1)}$ (see Fig. 4 in *Part II*). Therefore, the assembling must consider this fact, as explained in *Parts II* and *III*.

Regarding thickness assembling, there are different cases to consider:

- *Case 1*
  It involves only the displacement degrees of freedom. This is, for example, the case encountered when the multilayer matrix $\mathbf{K}_{u_x u_y}$ is generated. The assembling must take into account the continuity of the displacements and their ESL description (see *Part III* for more details).
- *Case 2*
  It involves displacement and out-of-plane stress degrees of freedom. This is, for example, the case encountered when the multilayer matrix $\mathbf{K}_{u_x s_z}$ is generated. The assembling must take into account the continuity of the displacements and their ESL description. In addition, the equilibrium of the transverse stresses must be enforced (see Fig. 5)
- *Case 3*
  It involves only the out-of-plane stress degrees of freedom. This is, for example, the case encountered when the multilayer matrix $\mathbf{K}_{s_y s_y}$ is generated. The assembling must take into account that the equilibrium of the transverse stresses must be enforced (see *Parts II* and *III*)

The pressure matrices are obtained from the pressure kernels and explained in *Part III*; therefore, the details are omitted. Regarding the pressure amplitudes, inputs of the problem (see Fig. 6), the difference between this case and the corresponding ESL case without the zig-zag terms should be noted. Fig. 7 shows the amplitude vectors for a particular case. The pressure amplitudes at multilayer level are *inputs* of the problem. Some input examples are shown in Fig. 6. Once the matrices are all assembled, the system of equations becomes (see *Part I*):

$$\begin{bmatrix} \mathbf{K}_{u_x u_x} & \mathbf{K}_{u_x u_y} & \mathbf{0}_{u_x u_z} & \mathbf{K}_{u_x s_x} & \mathbf{0}_{u_x s_y} & \mathbf{K}_{u_x s_z} \\ & \mathbf{K}_{u_y u_y} & \mathbf{0}_{u_y u_z} & \mathbf{0}_{u_y s_x} & \mathbf{K}_{u_y s_y} & \mathbf{K}_{u_y s_z} \\ & & \mathbf{0}_{u_z u_z} & \mathbf{K}_{u_z s_x} & \mathbf{K}_{u_z s_y} & \mathbf{K}_{u_z s_z} \\ & & & \mathbf{K}_{s_x s_x} & \mathbf{0}_{s_x s_y} & \mathbf{0}_{s_x s_z} \\ \text{Symm} & & & & \mathbf{K}_{s_y s_y} & \mathbf{0}_{s_y s_z} \\ & & & & & \mathbf{K}_{s_z s_z} \end{bmatrix} \cdot \begin{bmatrix} {}^x\mathbf{U} \\ {}^y\mathbf{U} \\ {}^z\mathbf{U} \\ {}^x\mathbf{S} \\ {}^y\mathbf{S} \\ {}^z\mathbf{S} \end{bmatrix} = \begin{bmatrix} {}^x\mathbf{R} \\ {}^y\mathbf{R} \\ {}^z\mathbf{R} \\ {}^x\mathbf{0} \\ {}^y\mathbf{0} \\ {}^z\mathbf{0} \end{bmatrix} \quad (17)$$

These expressions are formally the same as the ones encountered in the layerwise case (*Part II*) or in the ESL case (*Part III*).

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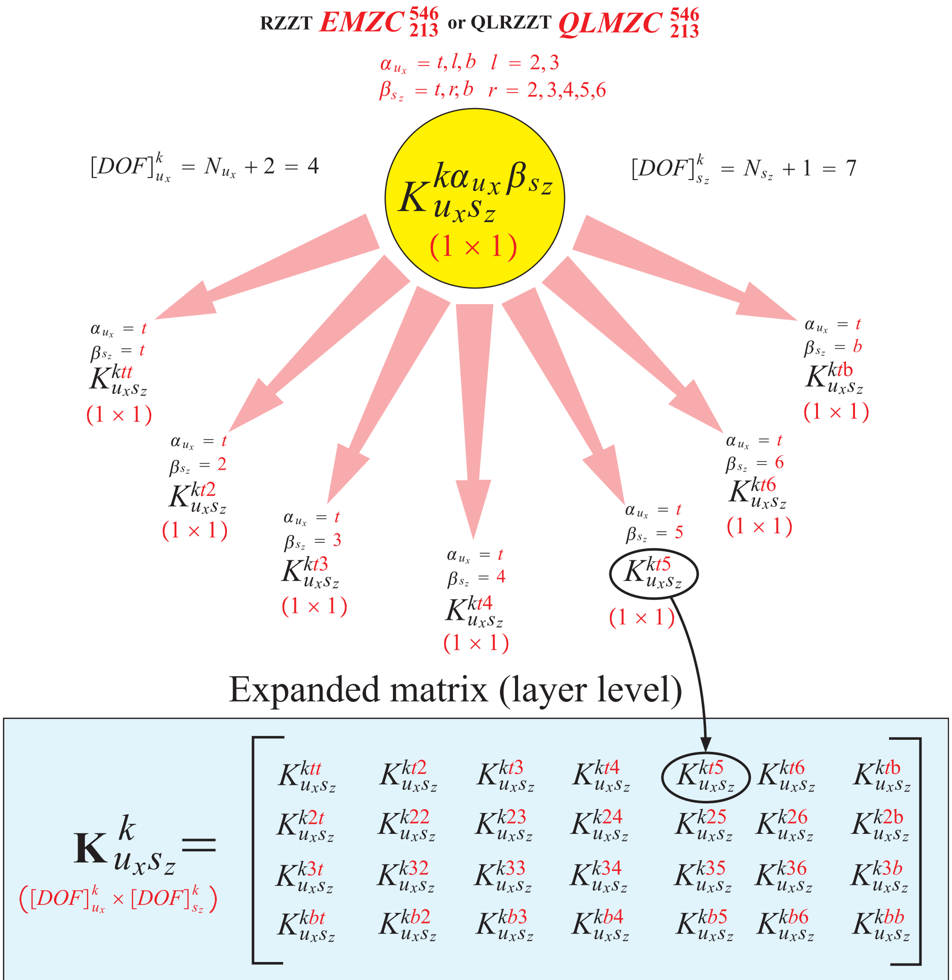

Fig. 4. Generalized unified formulation: example of expansion from a kernel to a layer matrix. The case of theories EMZC_{213}^{546} and QLMZC_{213}^{546}. From $K_{u_x s_z}^{k\alpha_{u_x}\beta_{s_z}}$ to $\mathbf{K}_{u_x s_z}^{k}$.

3. Example: a multilayered plate

As for the layerwise theories (*Part II*) and ESL theories (*Part III*), how the theories are created is shown. In particular, consider a rectangular multilayered plate consists of two layers. Let the thickness of the plate be h. The bottom layer has thickness $h_{k=1} = \frac{4}{7}h$. The top layer has thickness $h_{k=2} = \frac{3}{7}h$. How the matrices are obtained from the kernels of the generalized unified formulation is shown by considering top layer matrix $\mathbf{K}_{u_x s_z}^{k=2}$. The *kernel* associated with this matrix (at layer level) is the following:

$$K_{u_x s_z}^{k=2\,\alpha_{u_x}\beta_{s_z}} = -\frac{m\pi}{a} Z_{13 u_x s_z}^{k=2\,\alpha_{u_x}\beta_{s_z}} = -\frac{m\pi}{a} C_{13}^{k=2} \int_{z_{\text{bot}_{k=2}}}^{z_{\text{top}_{k=2}}} {}^{x}F_{\alpha_{u_x}}(z)\, {}^{z}\mathcal{F}_{\alpha_{s_z}}(z)\,\mathrm{d}z \tag{18}$$

where $z_{\text{bot}_{k=2}}$ is the z coordinate of the bottom surface of layer $k = 2$ (top layer); $z_{\text{top}_{k=2}}$ is the z coordinate of the top surface of layer $k = 2$. The reference plane is the middle plane of the whole plate. Thus, for the top layer

$$z_{\text{bot}_{k=2}} = \frac{h}{14} \qquad z_{\text{top}_{k=2}} = \frac{h}{2} \tag{19}$$

The expression of the kernel is then

$$K_{u_x s_z}^{k=2\,\alpha_{u_x}\beta_{s_z}} = -\frac{m\pi}{a} C_{13}^{k=2} \int_{\frac{h}{14}}^{\frac{h}{2}} {}^{x}F_{\alpha_{u_x}}(z)\, {}^{z}\mathcal{F}_{\alpha_{s_z}}(z)\,\mathrm{d}z \qquad \frac{h}{14} \leqslant z \leqslant \frac{h}{2} \tag{20}$$

Theory EMZC_{213}^{546} and, in particular, the term in which $\alpha_{u_x} = b$ and $\beta_{s_z} = 3$ are considered. In this case Eq. (20) becomes

$$K_{u_x s_z}^{k=2\,b3} = -\frac{m\pi}{a} C_{13}^{k=2} \int_{\frac{h}{14}}^{\frac{h}{2}} {}^{x}F_{b}(z)\, {}^{z}\mathcal{F}_{3}(z)\,\mathrm{d}z \tag{21}$$

In practice it is more convenient to transform the variables (see Eq. (4)) and numerically integrate using Gauss's quadrature formula in the interval [−1, +1]. However, since the goal is to show the procedure, we will simply use the physical coordinate z and write

$$\begin{aligned} {}^{x}F_b(z) &= (-1)^{k=2}\,\zeta_{k=2} = (-1)^2\,\zeta_{k=2} = \zeta_{k=2} \\ {}^{z}\mathcal{F}_3(z) &= P_3(z) - P_1(z) = \tfrac{5(\zeta_{k=2})^3 - 3\zeta_{k=2}}{2} - \zeta_{k=2} \end{aligned} \tag{22}$$

where

$$\zeta_{k=2} = \frac{2}{z_{\text{top}_{k=2}} - z_{\text{bot}_{k=2}}} z - \frac{z_{\text{top}_{k=2}} + z_{\text{bot}_{k=2}}}{z_{\text{top}_{k=2}} - z_{\text{bot}_{k=2}}} \tag{23}$$

Using Eqs. (19), (22) and (23):

$$\begin{aligned} {}^{x}F_b(z) &= \frac{2}{3}\left(\frac{7z}{h} - 2\right) \\ {}^{z}\mathcal{F}_3(z) &= -\frac{35\left(2h^3 - 39zh^2 + 168z^2h - 196z^3\right)}{27h^3} \end{aligned} \tag{24}$$

Substituting into Eq. (21) and calculating the integral:

$$K_{u_x s_z}^{k=2\,b3} = \frac{hm\pi}{7a} C_{13}^{k=2} \tag{25}$$

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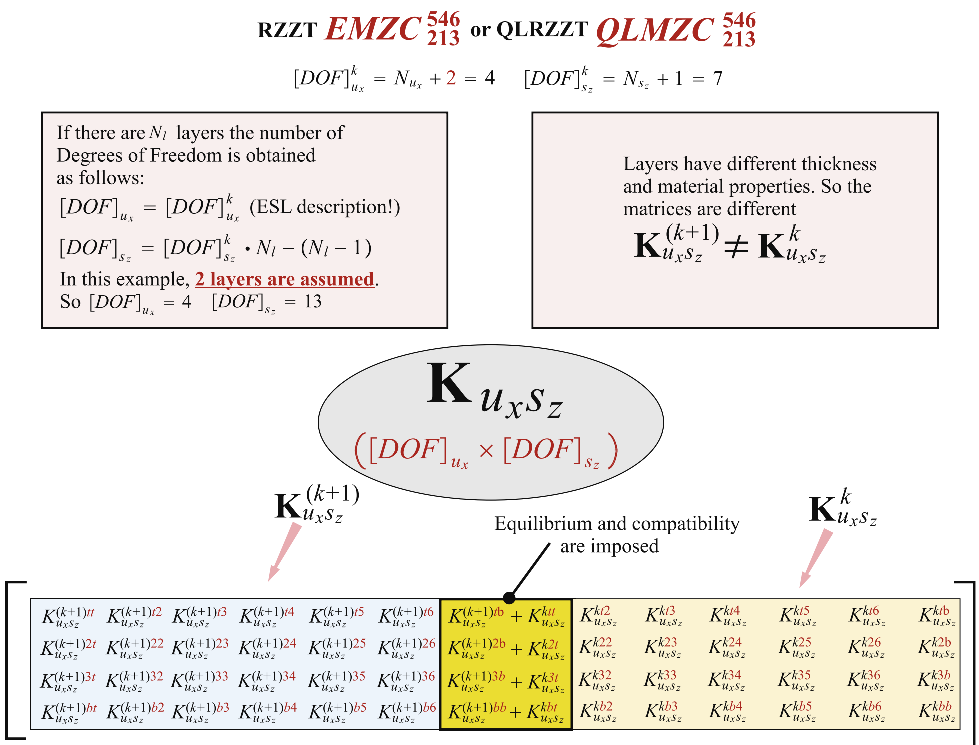

Fig. 5. Generalized unified formulation: example of assembling from layer matrices to multilayer matrix. Case of theory $EMZC^{546}_{213}$ or $QLMZC^{546}_{213}$. From $\mathbf{K}^{k}_{u_x s_z}$ and $\mathbf{K}^{(k+1)}_{u_x s_z}$ to $\mathbf{K}_{u_x s_z}$.

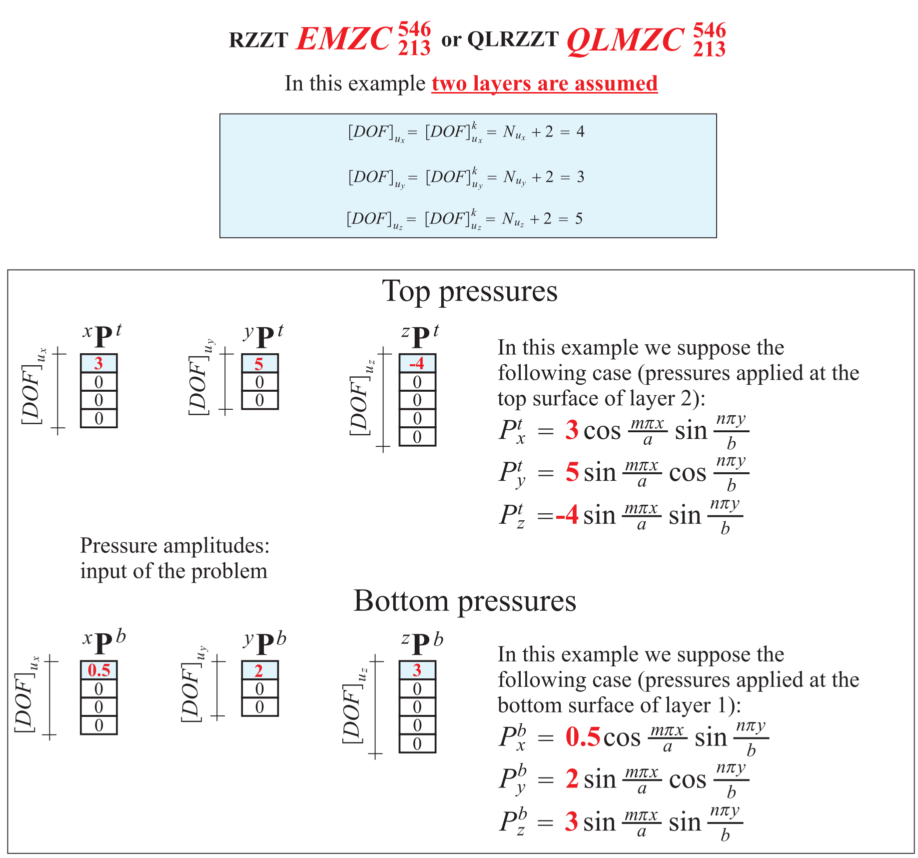

Fig. 6. Theory $EMZC^{546}_{213}$ or $QLMZC^{546}_{213}$. Example of pressure amplitudes and inputs of problem at multilayer level in two-layered case.

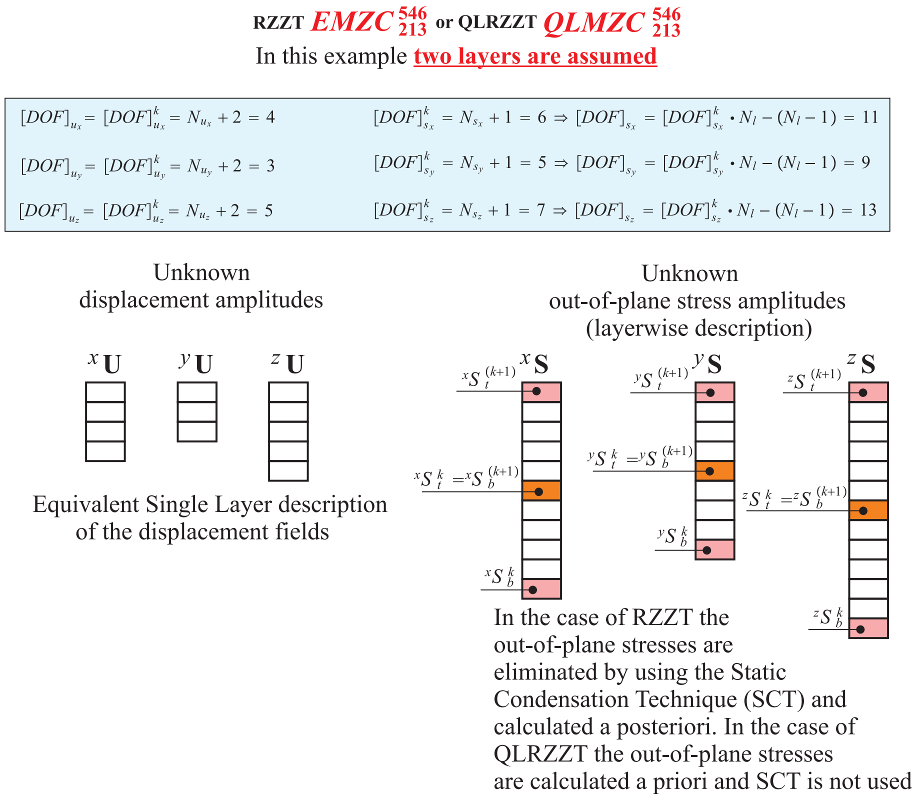

Fig. 7. Theory $EMZC_{213}^{546}$ or $QLMZC_{213}^{546}$. Multilayer unknown displacement and out-of-plane stress vectors in two-layered case.

Using the same procedure, it is possible to demonstrate that the top layer (with the above mentioned data) has the matrices reported in Appendix A. The pressure matrices are obtained using the definitions reported in *Part I*. For example, for the top layer ($k = 2$):

$$ {}^tD_{u_xu_x}^{k=2\,\alpha_{u_x}\beta_{u_x}} = {}^xF_{\alpha_{u_x}}^t\,{}^xF_{\beta_{u_x}}^t = {}^xF_{\alpha_{u_x}}\left(z = +\frac{h}{2}\right){}^xF_{\beta_{u_x}}\left(z = +\frac{h}{2}\right) \tag{26}$$

Therefore, the pressure matrices are the following:

$$ {}^t\mathbf{D}_{u_xu_x}^{k=2} = \begin{bmatrix} 1 & \frac{h}{2} & \frac{h^2}{4} & 1 \\ \frac{h}{2} & \frac{h^2}{4} & \frac{h^3}{8} & \frac{h}{2} \\ \frac{h^2}{4} & \frac{h^3}{8} & \frac{h^4}{16} & \frac{h^2}{4} \\ 1 & \frac{h}{2} & \frac{h^2}{4} & 1 \end{bmatrix} \quad {}^t\mathbf{D}_{u_yu_y}^{k=2} = \begin{bmatrix} 1 & \frac{h}{2} & 1 \\ \frac{h}{2} & \frac{h^2}{4} & \frac{h}{2} \\ 1 & \frac{h}{2} & 1 \end{bmatrix} \tag{27}$$

$$ {}^t\mathbf{D}_{u_zu_z}^{k=2} = \begin{bmatrix} 1 & \frac{h}{2} & \frac{h^2}{4} & \frac{h^3}{8} & 1 \\ \frac{h}{2} & \frac{h^2}{4} & \frac{h^3}{8} & \frac{h^4}{16} & \frac{h}{2} \\ \frac{h^2}{4} & \frac{h^3}{8} & \frac{h^4}{16} & \frac{h^5}{32} & \frac{h^2}{4} \\ \frac{h^3}{8} & \frac{h^4}{16} & \frac{h^5}{32} & \frac{h^6}{64} & \frac{h^3}{8} \\ 1 & \frac{h}{2} & \frac{h^2}{4} & \frac{h^3}{8} & 1 \end{bmatrix} \tag{28}$$

Now consider the first layer where $k = 1$. The coordinates of the bottom and top surfaces of the first layer are

$$ z_{\text{bot}_{k=1}} = -\frac{h}{2} \qquad z_{\text{top}_{k=1}} = \frac{h}{14} \tag{29}$$

For brevity only the matrices $\mathbf{K}_{u_xu_y}^{k=1}, \mathbf{K}_{u_xs_z}^{k=1}$ and $\mathbf{K}_{s_zs_z}^{k=1}$ are presented in Appendix B. To complete this example, consider a numerical case with the following data (numbers are chosen only to create the numerical example):

$$ m = 2; \quad n = 3; \quad a = 10; \quad b = 15 \quad h = 7 \tag{30}$$

Assume that the following materials are used:

$$ \text{Top layer} \begin{cases} E_{11}^{k=2} = 25 & E_{22}^{k=2} = 4 & E_{33}^{k=2} = 3 \\ G_{12}^{k=2} = \frac{1}{2} & G_{13}^{k=2} = \frac{3}{5} & G_{23}^{k=2} = \frac{1}{5} \\ \upsilon_{12}^{k=2} = \frac{1}{4} & \upsilon_{13}^{k=2} = \frac{27}{100} & \upsilon_{23}^{k=2} = \frac{29}{100} \quad h_{k=2} = \frac{3}{7}h \end{cases} \tag{31}$$

$$ \text{Bottom layer} \begin{cases} E_{11}^{k=1} = 20 & E_{22}^{k=1} = 5 & E_{33}^{k=1} = 4 \\ G_{12}^{k=1} = \frac{1}{2} & G_{13}^{k=1} = \frac{3}{5} & G_{23}^{k=1} = \frac{1}{5} \\ \upsilon_{12}^{k=1} = \frac{1}{4} & \upsilon_{13}^{k=1} = \frac{27}{100} & \upsilon_{23}^{k=1} = \frac{29}{100} \quad h_{k=1} = \frac{4}{7}h \end{cases} \tag{32}$$

$\vartheta = 0$ for both layers. The numerical values for the matrices $\mathbf{K}_{u_xu_y}^{k=1}, \mathbf{K}_{u_xs_z}^{k=1}, \mathbf{K}_{s_zs_z}^{k=1}, \mathbf{K}_{u_xu_y}^{k=2}, \mathbf{K}_{u_xs_z}^{k=2}$ and $\mathbf{K}_{s_zs_z}^{k=2}$ are reported in Appendix C.

4. Conclusion

For the first time in the literature, the extension of the generalized unified formulation to the cases of mixed variational statements (in particular Reissner's mixed variational theorem) and higher order zig-zag theories is presented. The displacements, which have an equivalent single layer description, are expanded along the thickness by using a Taylor series. The zig-zag form of the displacements is imposed by using Murakami's zig-zag function. The stresses σ_{zx}, σ_{zy} and σ_{zz} have a layerwise description and are expanded along the thickness of each layer by using Legendre polynomials. Each variable can be treated separately from the others. This allows the writing, with a single formal derivation and software, of ∞^6 mixed higher order zig-zag theories. If the stresses are eliminated using the static condensation technique the resulting theory is formally identical to a "classical" displacement-based higher order zig-zag theory. If the stresses are not eliminated then a quasi-layerwise model is obtained.

The new methodology based on the use of the generalized unified formulation allows the user to freely change the orders used for the expansion of the unknowns and to experiment the best combination that better approximates the structural problem under investigation. The compatibility of the displacements and the equilibrium between two adjacent layers enforced are a priori. *All* the theories are generated by expanding 1×1 matrices (the *kernels* of the generalized unified formulation), which are invariant with respect to the theory. Thus, with only 13 matrices (the kernels) ∞^6 theories can be generated without difficulties. These kernels are the same as the ones used in the layerwise and equivalent single layer cases discussed in *Parts II* and *III*.

The numerical performances and properties of mixed higher order zig-zag theories will be discussed in *Part V* (see [38]) of the present work.

$$\mathbf{K}_{u_y u_y}^{k=2} = \frac{(C_{66}^{k=2} b^2 m^2 + C_{22}^{k=2} a^2 n^2)\pi^2}{a^2 b^2} \begin{bmatrix} \frac{3}{7}h & \frac{6}{49}h^2 & 0 \\ \frac{6}{49}h^2 & \frac{57}{1372}h^3 & \frac{3}{98}h^2 \\ 0 & \frac{3}{98}h^2 & \frac{1}{7}h \end{bmatrix} \quad (37)$$

$$\mathbf{K}_{u_y s_y}^{k=2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{3}{14}h & -\frac{3}{7}h & 0 & 0 & \frac{3}{14}h \\ 1 & -2 & 0 & 0 & 1 \end{bmatrix} \quad (38)$$

$$\mathbf{K}_{u_y s_z}^{k=2} = \frac{C_{23}^{k=2} n\pi}{b} \begin{bmatrix} -\frac{3}{14}h & \frac{3}{7}h & 0 & 0 & 0 & 0 & -\frac{3}{14}h \\ -\frac{15}{196}h^2 & \frac{6}{49}h^2 & \frac{3}{98}h^2 & 0 & 0 & 0 & -\frac{9}{196}h^2 \\ -\frac{1}{14}h & 0 & \frac{1}{7}h & 0 & 0 & 0 & \frac{1}{14}h \end{bmatrix} \quad (39)$$

$$\mathbf{K}_{u_z s_x}^{k=2} = \frac{m\pi}{a} \begin{bmatrix} \frac{3}{14}h & -\frac{3}{7}h & 0 & 0 & 0 & \frac{3}{14}h \\ \frac{15}{196}h^2 & -\frac{6}{49}h^2 & -\frac{3}{98}h^2 & 0 & 0 & \frac{9}{196}h^2 \\ \frac{81}{2744}h^3 & -\frac{267}{6860}h^3 & -\frac{6}{343}h^3 & -\frac{9}{3430}h^3 & 0 & \frac{33}{2744}h^3 \\ \frac{2301}{192080}h^4 & -\frac{321}{24010}h^4 & -\frac{1089}{134456}h^4 & -\frac{27}{12005}h^4 & -\frac{81}{336140}h^4 & \frac{699}{192080}h^4 \\ \frac{1}{14}h & 0 & -\frac{1}{7}h & 0 & 0 & -\frac{1}{14}h \end{bmatrix} \quad (40)$$

Acknowledgement

The author thanks his sister Demasi Paola who inspired him with her strong will.

Appendix A. Explicit expressions of the layer matrices for a particular case

This appendix shows some of the matrices for the *top layer* ($k = 2$) of the structure described in Section 3. Theory EMZC_{213}^{546} is considered.

$$\mathbf{K}_{u_x u_x}^{k=2} = \frac{\pi^2 (C_{11}^{k=2} b^2 m^2 + C_{66}^{k=2} a^2 n^2)}{a^2 b^2} \begin{bmatrix} \frac{3}{7}h & \frac{6}{49}h^2 & \frac{57}{1372}h^3 & 0 \\ \frac{6}{49}h^2 & \frac{57}{1372}h^3 & \frac{75}{4802}h^4 & \frac{3}{98}h^2 \\ \frac{57}{1372}h^3 & \frac{75}{4802}h^4 & \frac{8403}{1344560}h^5 & \frac{6}{343}h^3 \\ 0 & \frac{3}{98}h^2 & \frac{6}{343}h^3 & \frac{1}{7}h \end{bmatrix} \quad (33)$$

$$\mathbf{K}_{u_x u_y}^{k=2} = \frac{(C_{12}^{k=2} + C_{66}^{k=2}) mn\pi^2}{ab} \begin{bmatrix} \frac{3}{7}h & \frac{6}{49}h^2 & 0 \\ \frac{6}{49}h^2 & \frac{57}{1372}h^3 & \frac{3}{98}h^2 \\ \frac{57}{1372}h^3 & \frac{75}{4802}h^4 & \frac{6}{343}h^3 \\ 0 & \frac{3}{98}h^2 & \frac{1}{7}h \end{bmatrix} \quad (34)$$

$$\mathbf{K}_{u_x s_x}^{k=2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ +\frac{3}{14}h & -\frac{3}{7}h & 0 & 0 & 0 & \frac{3}{14}h \\ +\frac{15}{98}h^2 & -\frac{12}{49}h^2 & -\frac{3}{49}h^2 & 0 & 0 & \frac{9}{98}h^2 \\ +1 & -2 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (35)$$

$$\mathbf{K}_{u_x s_z}^{k=2} = \frac{C_{13}^{k=2} m\pi}{a} \begin{bmatrix} -\frac{3}{14}h & \frac{3}{7}h & 0 & 0 & 0 & 0 & -\frac{3}{14}h \\ -\frac{15}{196}h^2 & \frac{6}{49}h^2 & \frac{3}{98}h^2 & 0 & 0 & 0 & -\frac{9}{196}h^2 \\ -\frac{81}{2744}h^3 & \frac{267}{6860}h^3 & \frac{6}{343}h^3 & \frac{9}{3430}h^3 & 0 & 0 & -\frac{33}{2744}h^3 \\ -\frac{1}{14}h & 0 & \frac{1}{7}h & 0 & 0 & 0 & +\frac{1}{14}h \end{bmatrix} \quad (36)$$

$$\mathbf{K}_{u_z s_y}^{k=2} = \frac{n\pi}{b} \begin{bmatrix} \frac{3}{14}h & -\frac{3}{7}h & 0 & 0 & \frac{3}{14}h \\ \frac{15}{196}h^2 & -\frac{6}{49}h^2 & -\frac{3}{98}h^2 & 0 & \frac{9}{196}h^2 \\ \frac{81}{2744}h^3 & -\frac{267}{6860}h^3 & -\frac{6}{343}h^3 & -\frac{9}{3430}h^3 & \frac{33}{2744}h^3 \\ \frac{2301}{192080}h^4 & -\frac{321}{24010}h^4 & -\frac{1089}{134456}h^4 & -\frac{27}{12005}h^4 & \frac{699}{192080}h^4 \\ \frac{1}{14}h & 0 & -\frac{1}{7}h & 0 & -\frac{1}{14}h \end{bmatrix} \quad (41)$$

$$\mathbf{K}_{u_z s_z}^{k=2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{14}h & -\frac{3}{7}h & 0 & 0 & 0 & 0 & \frac{3}{14}h \\ \frac{15}{98}h^2 & -\frac{12}{49}h^2 & -\frac{3}{49}h^2 & 0 & 0 & 0 & \frac{9}{98}h^2 \\ \frac{243}{2744}h^3 & -\frac{801}{6860}h^3 & -\frac{18}{343}h^3 & -\frac{27}{3430}h^3 & 0 & 0 & \frac{99}{2744}h^3 \\ 1 & -2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (42)$$

$$\mathbf{K}_{s_x s_x}^{k=2} = C_{55}^{k=2} h \begin{bmatrix} -\frac{1}{7} & +\frac{3}{14} & +\frac{1}{14} & 0 & 0 & -\frac{1}{14} \\ +\frac{3}{14} & -\frac{18}{35} & 0 & +\frac{3}{35} & 0 & +\frac{3}{14} \\ +\frac{1}{14} & 0 & -\frac{10}{49} & 0 & +\frac{3}{49} & -\frac{1}{14} \\ 0 & +\frac{3}{35} & 0 & -\frac{2}{15} & 0 & 0 \\ 0 & 0 & +\frac{3}{49} & 0 & -\frac{54}{539} & 0 \\ -\frac{1}{14} & +\frac{3}{14} & -\frac{1}{14} & 0 & 0 & -\frac{1}{7} \end{bmatrix} \quad (43)$$

$$\mathbf{K}_{s_y s_y}^{k=2} = C_{44}^{k=2} h \begin{bmatrix} -\frac{1}{7} & +\frac{3}{14} & +\frac{1}{14} & 0 & -\frac{1}{14} \\ +\frac{3}{14} & -\frac{18}{35} & 0 & +\frac{3}{35} & +\frac{3}{14} \\ +\frac{1}{14} & 0 & -\frac{10}{49} & 0 & -\frac{1}{14} \\ 0 & +\frac{3}{35} & 0 & -\frac{2}{15} & 0 \\ -\frac{1}{14} & +\frac{3}{14} & -\frac{1}{14} & 0 & -\frac{1}{7} \end{bmatrix} \quad (44)$$

$$\mathbf{K}_{s_z s_z}^{k=2} = C_{33}^{k=2} h \begin{bmatrix} -\frac{1}{7} & +\frac{3}{14} & +\frac{1}{14} & 0 & 0 & 0 & -\frac{1}{14} \\ +\frac{3}{14} & -\frac{18}{35} & 0 & +\frac{3}{35} & 0 & 0 & +\frac{3}{14} \\ +\frac{1}{14} & 0 & -\frac{10}{49} & 0 & +\frac{3}{49} & 0 & -\frac{1}{14} \\ 0 & +\frac{3}{35} & 0 & -\frac{2}{15} & 0 & +\frac{1}{21} & 0 \\ 0 & 0 & +\frac{3}{49} & 0 & -\frac{54}{539} & 0 & 0 \\ 0 & 0 & 0 & +\frac{1}{21} & 0 & -\frac{22}{273} & 0 \\ -\frac{1}{14} & +\frac{3}{14} & -\frac{1}{14} & 0 & 0 & 0 & -\frac{1}{7} \end{bmatrix} \quad (45)$$

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Appendix B. Explicit expressions of the layer matrices for a particular case

This appendix shows some of the matrices for the *bottom layer* ($k = 1$) of the structure described in Section 3. Theory EMZC_{213}^{546} is considered.

$$\mathbf{K}_{u_x u_y}^{k=1} = \frac{(C_{12}^{k=1} + C_{66}^{k=1})mn\pi^2}{ab}\begin{bmatrix} +\frac{4}{7}h & -\frac{6}{49}h^2 & 0 \\ -\frac{6}{49}h^2 & +\frac{43}{1029}h^3 & -\frac{8}{147}h^2 \\ +\frac{43}{1029}h^3 & -\frac{75}{4802}h^4 & +\frac{8}{343}h^3 \\ 0 & -\frac{8}{147}h^2 & +\frac{4}{21}h \end{bmatrix} \tag{46}$$

$$\mathbf{K}_{u_x s_z}^{k=1} = \frac{C_{13}^{k=1}m\pi}{a}\begin{bmatrix} -\frac{2}{7}h & +\frac{4}{7}h & 0 & 0 & 0 & 0 & -\frac{2}{7}h \\ +\frac{5}{147}h^2 & -\frac{6}{49}h^2 & +\frac{8}{147}h^2 & 0 & 0 & 0 & +\frac{13}{147}h^2 \\ -\frac{19}{2058}h^3 & +\frac{61}{1715}h^3 & -\frac{8}{343}h^3 & +\frac{32}{5145}h^3 & 0 & 0 & -\frac{67}{2058}h^3 \\ +\frac{2}{21}h & 0 & -\frac{4}{21}h & 0 & 0 & 0 & -\frac{2}{21}h \end{bmatrix} \tag{47}$$

$$\mathbf{K}_{s_z s_z}^{k=1} = C_{33}^{k=1}h\begin{bmatrix} -\frac{4}{21} & +\frac{2}{7} & +\frac{2}{21} & 0 & 0 & 0 & -\frac{2}{21} \\ +\frac{2}{7} & -\frac{24}{35} & 0 & +\frac{4}{35} & 0 & 0 & +\frac{2}{7} \\ +\frac{2}{21} & 0 & -\frac{40}{147} & 0 & +\frac{4}{49} & 0 & -\frac{2}{21} \\ 0 & +\frac{4}{35} & 0 & -\frac{8}{45} & 0 & +\frac{4}{63} & 0 \\ 0 & 0 & +\frac{4}{49} & 0 & -\frac{72}{539} & 0 & 0 \\ 0 & 0 & 0 & +\frac{4}{63} & 0 & -\frac{88}{819} & 0 \\ -\frac{2}{21} & +\frac{2}{7} & -\frac{2}{21} & 0 & 0 & 0 & -\frac{4}{21} \end{bmatrix} \tag{48}$$

Appendix C. Numeric expressions of the layer matrices for a particular case

This appendix shows some of the matrices for the *top layer* ($k = 2$) and *bottom layer* of the structure described in Section 3. Theory EMZC_{213}^{546} is considered.

$$\mathbf{K}_{u_x u_y}^{k=1} = \begin{bmatrix} 2.79 & -4.19 & 0 \\ -4.19 & 10.01 & -1.86 \\ 10.01 & -26.20 & 5.59 \\ 0 & -1.86 & 0.93 \end{bmatrix} \tag{49}$$

$$\mathbf{K}_{u_x s_z}^{k=1} = \begin{bmatrix} -0.44 & 0.87 & 0 & 0 & 0 & 0 & -0.44 \\ 0.36 & -1.31 & 0.58 & 0 & 0 & 0 & 0.95 \\ -0.69 & 2.67 & -1.75 & 0.47 & 0 & 0 & -2.44 \\ 0.15 & 0 & -0.29 & 0 & 0 & 0 & -0.15 \end{bmatrix} \tag{50}$$

$$\mathbf{K}_{s_z s_z}^{k=1} = \begin{bmatrix} -0.30 & 0.45 & 0.15 & 0 & 0 & 0 & -0.15 \\ 0.45 & -1.09 & 0 & 0.18 & 0 & 0 & 0.45 \\ 0.15 & 0 & -0.43 & 0 & 0.13 & 0 & -0.15 \\ 0 & 0.18 & 0 & -0.28 & 0 & 0.10 & 0 \\ 0 & 0 & 0.13 & 0 & -0.21 & 0 & 0 \\ 0 & 0 & 0 & 0.10 & 0 & -0.17 & 0 \\ -0.15 & 0.45 & -0.15 & 0 & 0 & 0 & -0.30 \end{bmatrix} \tag{51}$$

$$\mathbf{K}_{u_x u_y}^{k=2} = \begin{bmatrix} 1.79 & 3.58 & 0 \\ 3.58 & 8.50 & 0.89 \\ 8.50 & 22.36 & 3.58 \\ 0 & 0.89 & 0.60 \end{bmatrix} \tag{52}$$

$$\mathbf{K}_{u_x s_z}^{k=2} = \begin{bmatrix} -0.33 & 0.65 & 0 & 0 & 0 & 0 & -0.33 \\ -0.82 & 1.30 & 0.33 & 0 & 0 & 0 & -0.49 \\ -2.20 & 2.90 & 1.30 & 0.20 & 0 & 0 & -0.90 \\ -0.11 & 0 & 0.22 & 0 & 0 & 0 & 0.11 \end{bmatrix} \tag{53}$$

$$\mathbf{K}_{s_z s_z}^{k=2} = \begin{bmatrix} -0.31 & 0.46 & 0.15 & 0 & 0 & 0 & -0.15 \\ 0.46 & -1.11 & 0 & 0.18 & 0 & 0 & 0.46 \\ 0.15 & 0 & -0.44 & 0 & 0.13 & 0 & -0.15 \\ 0 & 0.18 & 0 & -0.29 & 0 & 0.10 & 0 \\ 0 & 0 & 0.13 & 0 & -0.22 & 0 & 0 \\ 0 & 0 & 0 & 0.10 & 0 & -0.17 & 0 \\ -0.15 & 0.46 & -0.15 & 0 & 0 & 0 & -0.31 \end{bmatrix} \tag{54}$$

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